1. Prove that a continuous real-valued function on a topological space that is zero on a dense subset must be the zero function.

   Notation: X is the space; D is the dense subset; f is the function.

   The singleton $\{0\}$ and its complement $\mathbf{R}^*$ partition $\mathbf{R}$, so their inverse images partition X. Since $D \subseteq f^{-1}(\{0\})$ (closed since f is continuous), $X = \overline{D} \subseteq f^{-1}(\{0\})$.

   Question: What does this say about continuous extensions of real-valued functions on a dense subset?

2. Given a family of topological spaces, pick a subset in each and prove that in general, the product of the subsets' closures is the closure of their product.

   Notation: $\{X_\alpha\}$ is the family of spaces; $X = \prod_\alpha X_\alpha$; $\pi_\alpha : X \to X_\alpha$ are the natural continuous projections; $\{E_\alpha \subseteq X_\alpha\}$ is the family of subsets.

   The box $\prod_\alpha \overline{E_\alpha} = X \setminus \bigcup_\alpha \pi_\alpha^{-1}(X_\alpha \setminus \overline{E_\alpha})$ is closed. Since $\prod_\alpha E_\alpha \subseteq \prod_\alpha \overline{E_\alpha}$, we have $\overline{\prod_\alpha E_\alpha} \subseteq \prod_\alpha \overline{E_\alpha}$.

   Alternate solution: Suppose $x \in \overline{\prod_\alpha E_\alpha}$. Then $\prod_\alpha E_\alpha$ has a net $x_\lambda \to x$. Since π_α is continuous, $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$, so $\pi_\alpha(x) \in \overline{E_\alpha}$, so $x \in \prod_\alpha \overline{E_\alpha}$.

   Conversely, let $x \in \prod_\alpha \overline{E_\alpha}$ and let $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ (U_{α_i} open in X_{α_i}) be a basic neighbourhood of x. Since $\pi_{\alpha_i}(x) \in \overline{E_{\alpha_i}}$, there exists $x_{\alpha_i} \in E_{\alpha_i} \cap U_{\alpha_i}$. Any x' such that $\pi_\alpha(x') \in E_\alpha$ in general, and in particular $\pi_{\alpha_i}(x') = x_{\alpha_i}$, is in $(\prod_\alpha E_\alpha) \cap U$. Thus, $x \in \overline{\prod_\alpha E_\alpha}$.

   Question: What happens if we replace the product topology on X with the box topology?

3. Suppose X is a topological space and $A \subseteq X$. Recall that A is a retract of X whenever there exists an onto continuous function $X \to A$ that is indentity on A.

   (a) Prove that A is a retract of X if and only if any continuous function on A can be extended to X.

   Let $f\colon X \to A$ be a retraction and suppose g is a continuous function on A. Then $g \circ f$ is an extension of g to X.

   Conversely, a continuous extension of the identity on A is a retraction.

   (b) Prove that if X is Hausdorff, then A must be closed in X.

   Suppose x_λ is a net in A convergent to x. Since the retraction f is continuous, $f(x_\lambda) = x_\lambda$ converges to $f(x)$ in A. Since X is Hausdorff, limits are unique, so $x = f(x)$, so $x \in A$.

   (c) Prove that the unit circle in the plane is a retract of the plane punctured at the origin.

   The map $f\colon \mathbf{C}^* \to S^1$ given by $f(z) = z/|z|$ is a retraction.

4. Given a point in a discrete space, which filters converge to that point? What happens in a trivial space?

   In the discrete topology $\{x\}$ is an open neighbourhood of x, so if $\mathcal{F} \to x$, we have $\{x\} \in \mathcal{F}$, so any superset of $\{x\}$ is in $\mathcal{F}$. Conversely, any element of $\mathcal{F}$ must meet $\{x\}$, so $\mathcal{F}$ is the ultrafilter of all subsets containing x.

   In the trivial topology the only open neighbourhood of x is the whole space, which belongs to every $\mathcal{F}$, so any $\mathcal{F} \to x$.

5. Prove that the intersection of compact subsets of a Hausdorff space is compact.

   Notation: X is the Hausdorff space; $\{K_\alpha \subseteq X\}$ is the family of compact subsets.

   Since X is Hausdorff, each K_α is closed, so $\bigcap_\alpha K_\alpha$ is closed. Since the latter is a closed subset of a compact space (pick any K_α), it is compact.

   Question: Can you come up with a counterexample if X is not Hausdorff?