Midterm 1 / 2011.10.12 / MAT 5243.001 / General Topology I

1. Prove that discs in a metric space are open.

   Suppose (X, d) is a metric space, $x \in X$, $\varepsilon > 0$, and let $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. Let $y \in B_\varepsilon(x)$. Let $\delta = \varepsilon - d(x, y)$. NB: $\delta > 0$. We want $B_\delta(y) \subseteq B_\varepsilon(x)$. Let $z \in B_\delta(y)$. Then $d(x, z) \leq d(y, z) + d(x, y) < \delta + d(x, y) = \varepsilon$. ◡̈

2. Prove that the topology induced by a metric restricted to a subspace of a metric space is the same as the subspace topology.

   Suppose (X, d) is a metric space and $Y \subseteq X$. Let $y \in Y$. Let $\varepsilon > 0$ and $B_\varepsilon(y) = \{x \in Y : d(x, y) < \varepsilon\}$. Since $B_\varepsilon(y) = \{x \in X : d(x, y) < \varepsilon\} \cap Y$ and $\{x \in X : d(x, y) < \varepsilon\}$ is open in X (see 1), $B_\varepsilon(y)$ is open in the subspace topology of Y. Conversely, let V be an open neighbourhood of y in the subspace topology of Y. Then $\exists$ open $U \subseteq X$ such that $V = U \cap Y$. Then $\exists \delta > 0$ such that $\{x \in X : d(x, y) < \delta\} \subseteq U$, so $B_\delta(y) = \{x \in X : d(x, y) < \delta\} \cap Y \subseteq U \cap Y = V$. ◡̈

3. State and prove a containment relation between the intersection of interiors and the interior of intersection. Show by way of an example that equality does not hold in general. Under what additional hypothesis can you expect equality?

   Proposition. Suppose X be a topological space and $\{E_\alpha\}_{\alpha\in A}$ is a family of subsets of X indexed by a set A. Then

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^\circ \subseteq \bigcap_{\alpha\in A} \mathring{E}_\alpha$$

   and if A is finite we have equality.

   Proof. For each $\beta \in A$

$$\bigcap_{\alpha\in A} E_\alpha \subseteq E_\beta$$

   Therefore, $\forall \beta \in A$

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^\circ \subseteq \mathring{E}_\beta \qquad \text{so} \qquad \left(\bigcap_{\alpha\in A} E_\alpha\right)^\circ \subseteq \bigcap_{\alpha\in A} \mathring{E}_\alpha$$

   Now suppose A is finite. Since $\forall \alpha \in A$ $\mathring{E}_\alpha \subseteq E_\alpha$,

$$\bigcap_{\alpha\in A} \mathring{E}_\alpha \subseteq \bigcap_{\alpha\in A} E_\alpha$$

   but a finite intersection of open sets is open, so

$$\bigcap_{\alpha\in A} \mathring{E}_\alpha \subseteq \left(\bigcap_{\alpha\in A} E_\alpha\right)^\circ$$

   Example. Let $E_n = \{x \in \mathbf{R} : x \neq 1/n\}$. Then each E_n is open in $\mathbf{R}$, so $\mathring{E}_n = E_n$, so

$$\bigcap_{n=1}^{\infty} \mathring{E}_n = \bigcap_{n=1}^{\infty} E_n = \{x \in \mathbf{R} : \forall n\ x \neq 1/n\}$$

   but 0 is in this set and not in its interior.

4. Given a subset of a topological space, show that its interior, the interior of its complement and its boundary (frontier) partition the whole space.

   Suppose X is a topological space and $Y \subseteq X$. By complementation $\overline{X \setminus Y} = X \setminus \overset{\circ}{Y}$, so $\partial Y = \overline{Y} \cap \overline{X \setminus Y} = \overline{Y} \cap (X \setminus \overset{\circ}{Y}) = \overline{Y} \setminus \overset{\circ}{Y}$, so $\overset{\circ}{Y}$ and ∂Y partition $\overline{Y}$. Again, by complementation $(X \setminus Y)^\circ = X \setminus \overline{Y}$, so we are done. ◡̈

5. Given a continuous map between topological spaces and a subset of the domain, state and prove a containment relation between the forward image of the closure of the subset and the closure of the forward image. Provide an example, where the two are not equal.

   Proposition. Suppose $f : X \to Y$ is continuous map between topological spaces and $E \subseteq X$. Then $f(\overline{E}) \subseteq \overline{f(E)}$.

   Proof. Since $f(E) \subseteq \overline{f(E)}$, $E \subseteq f^{-1}(\overline{f(E)})$. Since f is continuous and $\overline{f(E)}$ is closed in Y, $f^{-1}(\overline{f(E)})$ is closed in X, so $\overline{E} \subseteq f^{-1}(\overline{f(E)})$, so $f(\overline{E}) \subseteq \overline{f(E)}$. ◡̈

   Examples. For simplicity let's assume $E = X$. If $f(X)$ is not closed in Y, we cannot expect equality. E.g. the constant map of the Sierpinski space to itself, whose image is the non-closed point. Another example is $f : \mathbf{N} \to \mathbf{R}$ given by $f(n) = 1/n$. Since $\mathbf{N}$ is discrete, f is continuous. However, $f(\mathbf{N})$ is not closed, since it misses 0, which is a limit point.