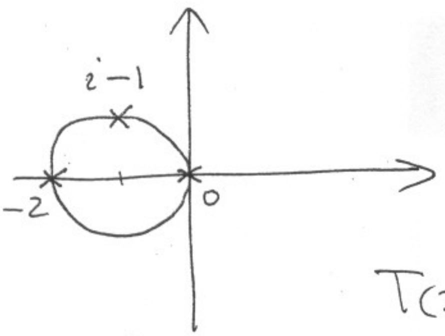
(1) First go the other way.
Pick 3 pts on the circle
& send them to $0, 1, \infty$.



$$0 \mapsto 0$$
$$i-1 \mapsto 1$$
$$-2 \mapsto \infty$$

$$T(z) = \frac{z}{z+2}\underbrace{\frac{i-1+2}{i-1}} = \frac{-iz}{z+2}$$

$$\frac{i+1}{i-1} = \frac{(i+1)^2}{-2} = -\frac{\cancel{-1}+2i+\cancel{1}}{2} = -i$$

Checkinsides: $T(-1) = \dfrac{i}{-1+2} = i$ ok.

Now compute inverse:

$$\begin{bmatrix} -i & 0 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{-2i}\begin{bmatrix} 2 & 0 \\ -1 & -i \end{bmatrix}$$

so $$T^{-1}(z) = \frac{2z}{-z-i} = \boxed{\frac{-2z}{z+i}}$$

This is not unique since I could have started By choosing different 3 pts. on the circle, e.g.
$$0 \mapsto 0$$
$$-2 \mapsto 1$$
$$-i-1 \mapsto \infty$$

2 (a) Let $z = x + iy$ $\quad f = u + iv$

$= u(x,y) + i\,v(x,y)$

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

C-R $\Rightarrow u_x = v_y \,,\ u_y = -v_x$

(Cauchy-Riemann Equations)

Let $g = \bar{f} = u - iv$

Then $$Dg = \begin{bmatrix} u_x & u_y \\ -v_x & -v_y \end{bmatrix}$$

CR $\Rightarrow u_x = -v_y \,,\ u_y = v_x$

$u_x = v_y \,,\ u_x = -v_y \Rightarrow v_y = -v_y \Rightarrow u_x = v_y = 0$

$u_y = -v_x \,,\ u_y = v_x = v_x = -v_x \Rightarrow u_y = v_x = 0$

$\therefore f = \text{const.}$

(b) Let $g(z) = f(\bar{z}) = u(x,-y) + i\,v(x,-y)$

$$Dg = \begin{bmatrix} u_x & -u_y \\ v_x & -v_y \end{bmatrix}$$

CR $\Rightarrow u_x = -v_y$

$u_y = v_x$

The rest of the argument is the same as in part (a)

(3) Let $u = re^{i\theta}$, $r \neq 0$.

Pick z & $w \neq 0$. Let $z = pe^{i\varphi}$, $w = qe^{i\psi}$.

The angle between z & w is $\psi - \varphi$

$f(z) = uz = re^{i\theta}pe^{i\varphi} = rpe^{i(\theta+\varphi)}$

$f(w) = rqe^{i(\theta+\psi)}$

The angle Between $f(z)$ & $f(w)$

is $(\theta + \psi) - (\theta + \varphi) = \psi - \varphi$ QED.

Linear algebra proof: Cosines of the angles between z & w and between $f(z)$ & $f(w)$ are $\dfrac{\langle z, w\rangle}{|z||w|}$ & $\dfrac{\langle f(z), f(w)\rangle}{|f(z)||f(w)|}$

where $\langle z, w\rangle$ is the usual dot product in $\mathbb{R}^2$,

In general $\langle z, w\rangle = \mathrm{Re}(\bar{z}w)$

[Proof: $\langle x+iy, a+ib\rangle = xa + yb$

$\bar{z}w = (x-iy)(a+ib) = xa + yb + i(xb - ya)$]

$\langle f(z), f(w)\rangle = \mathrm{Re}(\overline{uz}\, uw) = \mathrm{Re}(\underbrace{\bar{u}u}_{\text{real}}\bar{z}w) = |u|^2 \mathrm{Re}(\bar{z}w) = |u|^2\langle z, w\rangle$

$\dfrac{\langle f(z), f(w)\rangle}{|f(z)||f(w)|} = \dfrac{|u|^2\langle z, w\rangle}{|uz||uw|} = \dfrac{\cancel{|u|^2}\langle z, w\rangle}{\cancel{|u|^2}|z||w|} = \dfrac{\langle z, w\rangle}{|z||w|}$ QED

Note: f is linear [Proof: $f(az+bw) = u(az+bw) = auz + buw = af(z) + bf(w)$]

so conformality @ 0 $\Rightarrow$ conformality everywhere.

Exercise: Prove it!

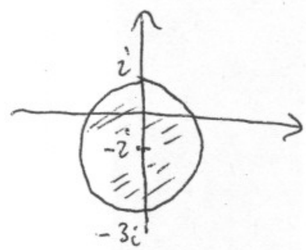
④ (a) Let's apply the ratio test.

$$\left| \frac{(z+i)^{n+1}}{2^{n+1}(n+1)^2} \Big/ \frac{(z+i)^n}{2^n n^2} \right| = |z+i| \frac{2^n n^2}{2^{n+1}(n+1)^2}$$

$$= |z+i| \frac{1}{2} \Big[\underbrace{\frac{n}{n+1}}_{\to 1} \Big]^2 \to |z+i| \frac{1}{2}$$

This is $< 1 \iff |z+i| < 2$ ← radius

Disc of convergence:



(b) If $|z+i| = 2$

$$\left| \sum \frac{(z+i)^n}{2^n n^2} \right| \le \sum \frac{|z+i|^n}{2^n n^2} = \sum \frac{\cancel{2^n}}{\cancel{2^n} n^2} = \sum \frac{1}{n^2} < \infty$$

By the integral test