

① For each $g \in G$, let $\phi_g \in \text{Inn}(G)$ be given by $\phi_g(x) = g x g^{-1}$.

The identity is $\phi_e \in \text{Inn}(G)$ \checkmark

$$\begin{aligned} \text{Given } \phi_g, \phi_h \in \text{Inn}(G), \phi_g \phi_h(x) &= g h x h^{-1} g^{-1} \\ &= g h x (gh)^{-1} = \phi_{gh}(x). \text{ Thus, } \phi_g \phi_h = \phi_{gh} \in \text{Inn}(G). \end{aligned}$$

$$\begin{aligned} \text{Since } \phi_g \phi_{g^{-1}}(x) &= g g^{-1} x g g^{-1} = x \text{ and} \\ \phi_{g^{-1}} \phi_g(x) &= g^{-1} g x g^{-1} g = x, \quad (\phi_g)^{-1} = \phi_{g^{-1}} \in \text{Inn } G \end{aligned}$$

Thus $\text{Inn}(G) < \text{Aut}(G)$

$$\begin{aligned} \text{Let } \psi \in \text{Aut}(G), \psi \phi_g \psi^{-1}(x) &= \psi(g \psi^{-1}(x) g^{-1}) \\ &= \psi(g) \times \psi(g^{-1}) = \psi(g) \times \psi(g)^{-1} = \phi_{\psi(g)}(x). \end{aligned}$$

Thus $\text{Inn}(G) \triangleleft \text{Aut}(G)$.

② Suppose $T \triangleleft H$. Let $x \in \phi^* T$, $y \in G$.

$$\text{Then } f(y x y^{-1}) = f(y) f(x) f(y)^{-1} \in T,$$

since $f(x) \in T$ and $T \triangleleft H$, so $y x y^{-1} \in \phi^* T$ \checkmark

Conversely, suppose $\phi^* T \triangleleft G$. Let $u \in T$, $v \in H$.

Since ϕ is surjective, $\exists x, y \in G$ $\phi(x) = u$, $\phi(y) = v$.

Since $x \in \phi^* T \triangleleft H$, $y x y^{-1} \in \phi^*(T)$, so

$$\phi(y x y^{-1}) = \phi(y) \phi(x) \phi(y)^{-1} = v u v^{-1} \in T. \quad \checkmark$$

Let $\pi: H \rightarrow \frac{H}{T}$ be the canonical projection.

$$\text{Then } \ker(\pi \phi) = \{x \in G : \pi \phi(x) = 0\}$$

$$= \{x \in G : \phi(x) \in T\} = \phi^* T. \text{ Since } \phi \text{ is onto,}$$

ϕ is $\pi \phi$, so by the 1st isomorphism thm $\frac{G}{\phi^* T} \cong \frac{H}{T}$ \checkmark

③ First, observe that $fi = 0$

Now let $g: C \rightarrow A$ be an R -mod morphism s.t. $fg = 0$.

$$\begin{array}{ccccc} \ker f & \xrightarrow{i} & A & \xrightarrow{f} & B \\ & \uparrow \exists! \phi & \uparrow g & & \\ & \exists! \phi & C & & \end{array}$$

Then $g_* C \subseteq \ker f$, so define

$\phi: C \rightarrow \ker f$ by $\phi(c) = g(c)$ for $c \in C$

Given $c \in C$ $i\phi(c) = ig(c) = g(c)$, so $i\phi = g$ and ϕ is unique.

$(i, \ker f)$ is universal for the subfunctor of $\text{Hom}_R(-, A)$ given by $\mathcal{F}(C) = \{g: C \rightarrow A : fg = 0\}$

④ First observe that $\pi f = 0$.

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\pi} & \frac{B}{f_* A} \\ & & \downarrow g & \swarrow \exists! \phi & \\ & & C & & \end{array}$$

Let $g: B \rightarrow C$ be an R -mod morphism such that $gf = 0$.

Then $g(f_* A) = 0$, so by the universal property of quotients $\exists! \phi$ s.t. $\phi\pi = g$

$(\pi, \frac{B}{f_* A})$ is universal for the subfunctor of $\text{Hom}_R(B, -)$ given by $\mathcal{F}(C) = \{g: B \rightarrow C : gf = 0\}$

⑤ First observe that $i: D \rightarrow Q(D)$ is injective.

Let $f: D \rightarrow F$ be an injective morphism to a field.

Let $[a, b] \in Q(D)$ (think $\frac{a}{b}$!) and define $\phi: Q(D) \rightarrow F$

by $\phi([a, b]) = f(a)f(b)^{-1}$. Note that $f(b) \neq 0$ in F ,

since $b \neq 0$ and f is injective. If $[a, b] = [a', b']$,

$ab' = a'b$, so $f(a)f(b') = f(a')f(b)$, so $f(a)f(b)^{-1} = f(a')f(b')^{-1}$

so ϕ is well-defined.

$$\begin{aligned}\phi([a, b][a', b']) &= \phi([aa', bb']) = f(aa')f(bb')^{-1} \\ &= f(a)f(b)^{-1}f(a')f(b')^{-1} = \phi([a, b])\phi([a', b'])\end{aligned}$$

$$\begin{aligned}\phi([a, b] + [a', b']) &= \phi([ab' + a'b, bb']) = \\ &= [f(a)f(b') + f(a')f(b)]f(b)^{-1}f(b')^{-1} = f(a)f(b)^{-1} + f(a')f(b')^{-1} \\ &= \phi([a, b]) + \phi([a', b'])\end{aligned}$$

$\therefore \phi$ is a ring hom

(Note that since $Q(F) \cong F$, ϕ is none other than $Q(f)$.)

$$\phi_i(a) = \phi([a, 1]) = f(a), \text{ so } \phi_i = f \quad \smile$$

Let $\phi': Q(D) \rightarrow F$ with $\phi'_i = f$. Given $[a, b] \in Q(D)$
 $\phi'([a, 1]) = \phi'_i(a) = f(a)$ and $\phi'([b, 1]) = \phi'_i(b) = f(b)$.

$$\begin{aligned}\text{Then } \phi'([a, b]) &= \phi'([a, 1][b, 1]^{-1}) = \phi'([a, 1])\phi'([b, 1])^{-1} \\ &= f(a)f(b)^{-1}, \text{ so } \phi \text{ is unique. } \quad \smile\end{aligned}$$

$(i, Q(D))$ is universal for subfunctor $\text{Hom}[D, -]$, restricted to fields, given by $\mathcal{F}(F) = \{f: D \rightarrow F: f \text{ is } 1-1\}$

⑥ $\mathbb{R}^{\mathbb{R}}$ is not an integral domain. E.g. let $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.
 Then $f(x)f(-x) = 0$.

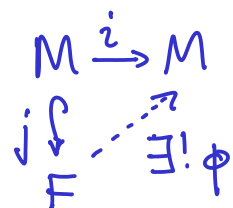
The units of $\mathbb{R}^{\mathbb{R}}$ are functions that are nowhere 0.

Suppose \mathcal{I} is a superideal of \mathcal{J} ($\mathcal{J} \subsetneq \mathcal{I}$). Let $f \in \mathcal{I} \setminus \mathcal{J}$.
 Then $f(c) \neq 0$. Let $g(x) = f(x) - f(c)$. Then $g(c) = 0$, so $g \in \mathcal{J}$,
 so the unit $f(c) = f(x) - g(x) \in \mathcal{I} \quad \smile$

Alt.: Define $\varepsilon: \mathbb{R}^{\mathbb{R}} \rightarrow \mathbb{R}$ by $\varepsilon(f) = f(c)$, show that ε is a ring morphism onto a field, so $\mathcal{J} = \ker \varepsilon$ is a max. ideal.

⑦ let $F = R^{(M)}$ be the free R -mod on the underlying set of M .

Since F is free, $\exists! \phi: F \rightarrow M$ with $\phi j = i$, where j is the canonical inclusion $M \hookrightarrow R^{(M)}$ and i is the identity function.

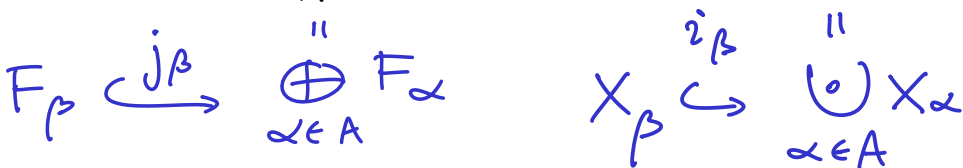


Since i is onto, so is ϕ .

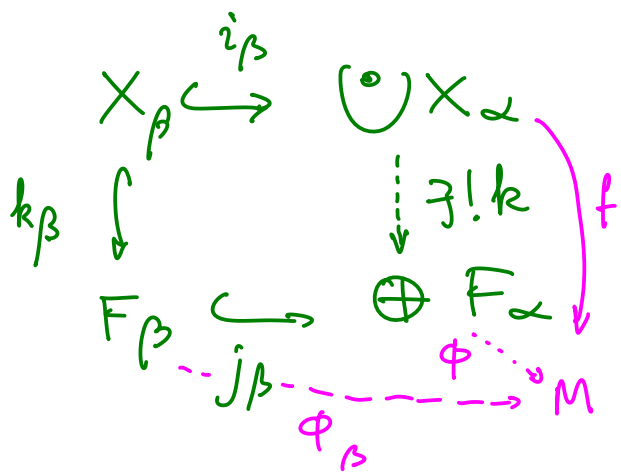
⑧ let $A \neq \emptyset$ (indexing set), F_α -free R -mod on X_α ($\alpha \in A$)

Notation: $k_\alpha: X_\alpha \hookrightarrow F_\alpha$

Claim $\coprod_{\alpha \in A} F_\alpha$ is free on $\coprod_{\alpha \in A} X_\alpha$



$$z \mapsto (\gamma \mapsto \begin{cases} z & \text{if } \delta = \beta \\ 0 & \text{otherwise} \end{cases})$$



By the univ. prop. of $\coprod X_\alpha$ $\exists! k: \cup X_\alpha \rightarrow \oplus F_\alpha$ s.t.

$$k i_\beta = j_\beta k_\beta$$

Claim: k is 1-1

Let $x \in X_\alpha, x' \in X_\beta$ s.t. $\underbrace{k i_\alpha x}_{j_\alpha k_\alpha x} = \underbrace{k i_\beta x'}_{j_\beta k_\beta x'}$

$$\gamma \mapsto \begin{cases} k_\alpha x & \text{if } \gamma = \alpha \\ 0 & \text{oth.} \end{cases} \quad \gamma \mapsto \begin{cases} k_\beta x' & \text{if } \gamma = \beta \\ 0 & \text{oth.} \end{cases}$$

$$\text{If } \alpha = \beta, \quad k_\alpha x = k_\alpha x', \quad x = x'$$

$$\text{If } \alpha \neq \beta, \quad \forall x \quad j_\alpha k_\alpha x = 0 \text{ or } j_\beta k_\beta x' = 0, \quad \& \quad x = 0, \quad x' = 0. \quad \ddot{\cup}$$

Since F_β is free on X_β , $\exists!$ $\phi_\beta: F_\beta \rightarrow M$ s.t. $f \circ i_\beta = \phi_\beta k_\beta$

By the univ. property of $\coprod F_\alpha$, $\exists!$ ϕ s.t. $\phi_\beta = \phi \circ j_\beta$

Claim: $f = \phi k$

Let $x \in \cup X_\alpha$. $\exists \beta \in A$ $x = i_\beta(x_\beta)$ for some $x_\beta \in X_\beta$

$$f(x) = f i_\beta x_\beta = \phi_\beta k_\beta x_\beta = \phi j_\beta k_\beta x_\beta = \phi k i_\beta x_\beta = \phi k(x) \quad \ddot{\cup}$$