MAT 5173 Algebra I Fall 2013

① For each $g \in G$, let $\phi_g \in \mathrm{Inn}(G)$ be given by $\phi_g(x) = gxg^{-1}$.

the identity is $\phi_e \in \mathrm{Inn}(G)$ ☺

Given $\phi_g, \phi_h \in \mathrm{Inn}(G)$, $\phi_g \phi_h(x) = ghxh^{-1}g^{-1} =$

$= ghx(gh)^{-1} = \phi_{gh}(x)$. Thus, $\phi_g\phi_h = \phi_{gh} \in \mathrm{Inn}(G)$.

Since $\phi_g \phi_{g^{-1}}(x) = gg^{-1}xgg^{-1} = x$ and

$\phi_{g^{-1}}\phi_g(x) = g^{-1}gxg^{-1}g = x$, $(\phi_g)^{-1} = \phi_{g^{-1}} \in \mathrm{Inn}\, G$

Thus $\mathrm{Inn}(G) < \mathrm{Aut}(G)$

Let $\psi \in \mathrm{Aut}(G)$, $\psi\phi_g\psi^{-1}(x) = \psi(g\psi^{-1}(x)g^{-1})$

$= \psi(g)x\psi(g^{-1}) = \psi(g)x\psi(g)^{-1} = \phi_{\psi(g)}(x)$.

Thus $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

② Suppose $T \lhd H$. Let $x \in \phi^*T$, $y \in G$.

Then $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} \in T$,

since $f(x) \in T$ and $T \lhd H$, so $yxy^{-1} \in \phi^*T$ ☺

Conversely, Suppose $\phi^*T \lhd G$. Let $u \in T$, $v \in H$.

Since ϕ is surjective, $\exists\, x, y \in G$ $\phi(x) = u$, $\phi(y) = v$.

Since $x \in \phi^*T \lhd H$, $yxy^{-1} \in \phi^*(T)$, so

$\phi(yxy^{-1}) = \phi(y)\phi(x)\phi(y)^{-1} = vuv^{-1} \in T$. ☺

Let $\pi : H \to \frac{H}{T}$ be the canonical projection.

Then $\ker(\pi\phi) = \{x \in G : \pi\phi(x) = 0\}$

$= \{x \in G : \phi(x) \in T\} = \phi^*T$. Since ϕ is onto,

so is $\pi\phi$, so by the 1st isomorphism thm $\frac{G}{\phi^*T} \cong \frac{H}{T}$ ☺

(3) First, observe that $fi = 0$

New let $g : C \to A$ be an R-mod morphism s.t. $fg = 0$.

$$\begin{tikzcd} \ker f \arrow[r, hook, "i"] & A \arrow[r, "f"] & B \\ & C \arrow[u, "g"'] \arrow[ul, dashed, "\exists!\phi"] & \end{tikzcd}$$

Then $g_* C \subseteq \ker f$, so define $\phi : C \to \ker f$ by $\phi(c) = g(c)$ for $c \in C$

Given $c \in C$ $\quad i\phi(c) = i\,g(c) = g(c)$, so $i\phi = g$ and ϕ is unique.

$(i, \ker f)$ is universal for the subfunctor of $\mathrm{Hom}_R(-, A)$ given by $\mathcal{F}(C) = \{ g : C \to A : fg = 0 \}$

(4) First observe that $\pi f = 0$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\pi"] \arrow[d, "g"'] & \frac{B}{f^* A} \arrow[dl, dashed, "\exists!\phi"] \\ & C & \end{tikzcd}$$

Let $g : B \to C$ be an R-mod morphism such that $gf = 0$.

Then $g(f_* A) = 0$, so by the universal property of quotients $\exists!\,\phi$ s.t. $\phi\pi = g$

$\left(\pi, \frac{B}{f^* A}\right)$ is universal for the subfunctor of $\mathrm{Hom}_R(B, -)$ given by $\mathcal{F}(C) = \{ g : B \to C : gf = 0 \}$

(5) First observe that $i : D \to Q(D)$ is injective.

Let $f : D \to F$ be an injective morphism to a field.

Let $[a, b] \in Q(D)$ (think $\frac{a}{b}$ ☺) and define $\phi : Q(D) \to F$ by $\phi([a,b]) = f(a) f(b)^{-1}$. Note that $f(b) \neq 0$ in F, since $b \neq 0$ and f is injective. If $[a,b] = [a', b']$, $ab' = a'b$, so $f(a) f(b') = f(a') f(b)$, so $f(a) f(b)^{-1} = f(a') f(b')^{-1}$ so ϕ is well-defined.

$$\phi([a,b][a',b']) = \phi([aa', bb']) = f(aa')f(bb')^{-1}$$
$$= f(a)f(b)^{-1}f(a')f(b')^{-1} = \phi([a,b])\,\phi([a',b'])$$

$$\phi([a,b] + [a',b']) = \phi([ab' + a'b, bb']) =$$
$$= \left[f(a)f(b') + f(a')f(b)\right] f(b)^{-1}f(b')^{-1} = f(a)f(b)^{-1} + f(a')f(b')^{-1}$$
$$= \phi([a,b]) + \phi([a',b'])$$

$\therefore$ ϕ is a ring hom

(Note that since $Q(F) \cong F$, ϕ is none other than $Q(f)$).

$\phi i(a) = \phi([a,1]) = f(a)$, so $\phi i = f$ ☺

Let $\phi' : Q(D) \to F$ with $\phi' i = f$. Given $[a,b] \in Q(D)$
$\phi'([a,1]) = \phi' i(a) = f(a)$ and $\phi'([b,1]) = \phi' i(b) = f(b)$.

Then $\phi'([a,b]) = \phi'([a,1][b,1]^{-1}) = \phi'([a,1])\,\phi'([b,1])^{-1}$
$= f(a)f(b)^{-1}$, so ϕ is unique. ☺

$(i, Q(D))$ is universal for subfunctor $\mathrm{Hom}[D,-]$, restricted to fields, given by $\mathcal{F}(F) = \{f : D \to F : f \text{ is } 1\text{-}1\}$

⑥ $\mathbb{R}^{\mathbb{R}}$ is <u>not</u> an integral domain. E.g. let $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.
Then $f(x)f(-x) = 0$.

The units of $\mathbb{R}^{\mathbb{R}}$ are functions that are nowhere 0.

Suppose I is a superideal of J ($J \subsetneq I$). Let $f \in I \setminus J$.
Then $f(c) \neq 0$. Let $g(x) = f(x) - f(c)$. Then $g(c) = 0$, so $g \in J$,
so the unit $f(c) = f(x) - g(x) \in I$ ☺

<u>Alt.</u>: Define $\varepsilon : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ by $\varepsilon(f) = f(c)$, show that ε is a ring morphism onto a field, so $J = \ker \varepsilon$ is a max. ideal.

(7) Let $F = R^{(M)}$ be the free R-mod on the underlying set of M.

Since F is free, $\exists!\, \phi : F \to M$ with $\phi j = i$, where j is the canonical inclusion $M \hookrightarrow R^{(M)}$ and i is the identity function.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, hook, "j"'] & M \\ F \arrow[ur, dashed, "\exists!\,\phi"'] & \end{tikzcd}$$

Since i is onto, so is ϕ.

(8) Let $A \neq \phi$ (indexing set), F_α - free R-mod on X_α $(\alpha \in A)$

<u>Notation:</u> $k_\alpha : X_\alpha \hookrightarrow F_\alpha$

<u>Claim</u> $\coprod\limits_{\alpha \in A} F_\alpha$ is free on $\coprod\limits_{\alpha \in A} X_\alpha$

$$F_\beta \xhookrightarrow{j_\beta} \underset{\alpha\in A}{\bigoplus} F_\alpha \qquad X_\beta \xhookrightarrow{i_\beta} \underset{\alpha\in A}{\dot\bigcup} X_\alpha$$

$$z \longmapsto \left(\gamma \mapsto \begin{cases} z & \text{if } \gamma = \beta \\ 0 & \text{otherwise} \end{cases}\right)$$

$$\begin{tikzcd} X_\beta \arrow[r, hook, "i_\beta"] \arrow[d, hook, "k_\beta"'] & \dot\bigcup X_\alpha \arrow[d, dashed, "\exists!\,k"] \arrow[dd, bend left, "f"] \\ F_\beta \arrow[r, hook] \arrow[dr, dashed, "j_\beta"'] \arrow[dr, dashed, bend right, "\phi_\beta"'] & \bigoplus F_\alpha \arrow[d, dotted, "\phi"] \\ & M \end{tikzcd}$$

By the univ. prop. of $\coprod X_\alpha$ $\exists!\, k : \dot\bigcup X_\alpha \to \bigoplus F_\alpha$ s.t.

$$k i_\beta = j_\beta k_\beta$$

<u>Claim</u>: k is 1-1

Let $x \in X_\alpha$, $x' \in X_\beta$ s.t. $\underbrace{k i_\alpha x}_{j_\alpha k_\alpha x} = \underbrace{k i_\beta x'}_{j_\beta k_\beta x'}$

$$\gamma \mapsto \begin{cases} k_\alpha x & \text{if } \gamma = \alpha \\ 0 & \text{oth.} \end{cases} \qquad \gamma \mapsto \begin{cases} k_\beta x' & \text{if } \gamma = \beta \\ 0 & \text{oth.} \end{cases}$$

If $\alpha=\beta$, $k_\alpha x = k_\alpha x'$, $x = x'$

If $\alpha \neq \beta$, $\forall \gamma$ $j_\alpha k_\alpha x = 0$ or $j_\beta k_\beta x' = 0$, so $x = 0$, $x' = 0$. ∵

Since F_β is free on X_β, $\exists! \, \phi_\beta : F_\beta \to M$ s.t. $f \circ i_\beta = \phi_\beta k_\beta$

By the univ. property of $\coprod F_\alpha$, $\exists! \, \phi$ s.t. $\phi_\beta = \phi \circ j_\beta$

<u>Claim:</u> $f = \phi k$

Let $x \in \dot{\bigcup} X_\alpha$. $\exists \beta \in A$ $x = i_\beta(x_\beta)$ for some $x_\beta \in X_\beta$

$f(x) = f i_\beta x_\beta = \phi_\beta k_\beta x_\beta = \phi j_\beta k_\beta x_\beta = \phi k i_\beta x_\beta = \phi k(x)$ ∵