1 (cf. II.13a, II.3.2, II.3.9) Since $\mathbf{Z}_4$ is generated by 1, an endomorphism f is uniquely determined by $f(1)$. The relation satisfied by 1 is $1+1+1+1=0$, so since $f(0)=0$ we must have $f(1)+f(1)+f(1)+f(1)=0$. This is automatically satisfied by any element of $\mathbf{Z}_4$. Thus, we may choose $f(1)$ to be any element of $\mathbf{Z}_4$. This means that there are 4 possible choices for $f(1)$ and, therefore, 4 endomorphisms of $\mathbf{Z}_4$.

In order for f to be an automorphism, $f(1)$ must be a generator of $\mathbf{Z}_4$. Since neither 0 nor 2 generate all of $\mathbf{Z}_4$, we must have $f(1)=1$ or $f(1)=3$. Thus, there are 2 automorphisms of $\mathbf{Z}_4$: the identity and the permutation $(1,3)$. Any group generated by an element of order 2 is isomorphic to $\mathbf{Z}_2$, so since $\mathrm{Aut}(\mathbf{Z}_4)$ is generated by the 2-cycle $(1,3)$, we have $\mathrm{Aut}(\mathbf{Z}_4)\cong\mathbf{Z}_2$.

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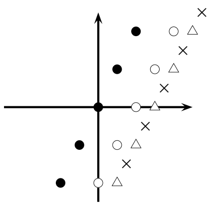
2 (cf. II.8.4) $H=\{(n,2n)\colon n\in\mathbf{Z}\}$ and its cosets are of the form $(x,y)+H=\{(n+x,2n+y)\colon n\in\mathbf{Z}\}$, where $(x,y)\in\mathbf{R}^2$.



- • H
- ○ $(1,-2)+H$
- △ $(3,0)+H$
- × $(0.5,1)+H$

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3 (a) Let $a\in\ker f$ and $x\in G$. Then $f(xax^{-1})=f(x)f(a)f(x^{-1})=f(x)f(x^{-1})=f(xx^{-1})=f(1)=1$, so $xax^{-1}\in\ker f$.

(b) Let $H=S_3$, $G=S_2$, and $f\colon S_2\to S_3$ the inclusion morphism.
The image of f contains $(1,2)$, but not $(2,3)(1,2)(2,3)^{-1}=(2,3)(1,2)(2,3)=(1,3)$.

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4 (cf. III.1.5) If $f\colon\mathbf{Z}_2\to\mathbf{Z}$ is a group morphism, then $f(1)=1$ and $0=f(0)=f(1+1)=f(1)+f(1)$, so $f(1)=0$, so $f=0$. Ring morphisms must preserve 1, so there are no ring morphisms and only the zero group morphism $\mathbf{Z}_2\to\mathbf{Z}$.

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5 (cf. III.7.3) If f is a ring morphism $\mathbf{Z}_2[x]\to\mathbf{Z}_2[x]$, we have $f(0)=0$ and $f(1)=1$, so f preserves constants.
If $\displaystyle\sum_{k=0}^{n}a_kx^k\in\mathbf{Z}_2[x]$, then $\displaystyle f\left[\sum_{k=0}^{n}a_kx^k\right]=\sum_{k=0}^{n}f(a_k)f(x^k)=\sum_{k=0}^{n}a_kf(x)^k$.
In other words, f is evaluation at $f(x)$. In order for f to be invertible, $f(x)$ must have degree 1. The only two polynomials of degree 1 in $\mathbf{Z}_2[x]$ are x and $x+1$. Therefore, there are 2 automorphisms of $\mathbf{Z}_2[x]$.

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6 (cf. III.13.10, III.7.5) A ring morphism $f\colon\mathbf{Q}[x]\to\mathbf{Q}$, must preserve constants (therefore onto) and is uniquely determined by $f(x)$. In light of the main theorem on quotient rings (domain/kernel $\cong$ image), it suffices to find f with kernel J. In particular, we want $0=f(x+1)=f(x)+1$, so choose $f(x)=-1$. In other words, let f be evaluation at -1, i.e. $f(a_0+a_1x+a_2x^2+...)=a_0-a_1+a_2-...$

To prove that $\ker f=J$ let $p\in J$. Then $p(x)=(x+1)q(x)$ for some $q\in\mathbf{Q}[x]$, so $f(p)=f((x+1)q)=f(x+1)f(q)=0\cdot f(q)=0$, so $p\in\ker f$. Conversely, suppose $p\in\ker f$. Then $f(p(x))=p(-1)=0$, so $x+1$ divides p [proof: by the division algorithm $p(x)=(x+1)q(x)+r$ for some $q(x)\in\mathbf{Q}[x]$ and $r\in\mathbf{Q}$; substituting $x=-1$ gives $r=0$], so $p\in J$.