Midterm

1. Determine whether $A = \{1/n : n = 1, 2...\}$ is an open, closed, both, or neither as a subset of the real line. Prove your assertion.

(i) A is not closed in $\mathbb{R}$: $0 \in A'$, but $0 \notin A$, so $A' \not\subseteq A$:)

(ii) A is not open in $\mathbb{R}$: A does not contain any open intervals.

2. Suppose X and Y are topological spaces and $f: X \to Y$. Prove that

   (a) if $U \subseteq Y$, then $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$

   (b) f is continuous on $X \Leftrightarrow \forall$ closed $V \subseteq Y$, $f^{-1}(V)$ is closed in X

a: $x \in f^{-1}(Y \setminus U) \Leftrightarrow f(x) \in Y \setminus U \Leftrightarrow f(x) \notin U$

$\Leftrightarrow x \notin f^{-1}(U) \Leftrightarrow x \in X \setminus f^{-1}(U)$:)

b: Recall: f is cont. on $X \Leftrightarrow \forall$ open $U \subseteq Y$, $f^{-1}(U)$ is open

[Notes 1/13 p.1]

$\Rightarrow$ If V is closed in Y, $Y \setminus V$ is open in Y,

so $f^{-1}(Y \setminus V)$ is open in X, so by 2a

$X \setminus f^{-1}(V)$ is open in X, so $f^{-1}(V)$ is closed

$\Leftarrow$ If $U \subseteq Y$ is open, then $Y \setminus U$ is closed in Y,

so $f^{-1}(Y \setminus U)$ is closed in X, so by 2a

$X \setminus f^{-1}(U)$ is closed in X, so $f^{-1}(U)$ is open in X

3. Suppose X is a topological space and $A \subseteq X$. Prove that the boundary of A is the intersection of the closures of A and its complement in X.

$\partial A = \bar{A} \cap \overline{A^c}$

(i) $a \in \bar{A} = A \cup A' \iff \forall$ open nbd. V of a, $V \cap A \neq \emptyset$

[Lemma, notes 2/21 p.1]

Pf: $V \cap A \neq \emptyset \iff a \in A$ or $(V \setminus \{a\}) \cap A \neq \emptyset$

(ii) $a \in \partial A \iff \forall$ open nbd. V of a $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$

$\iff a \in \bar{A}$ and $a \in \overline{A^c}$:)

Slick proof: $\partial A = \bar{A} \setminus \mathring{A} = \bar{A} \cap \mathring{A}^c = \bar{A} \cap \overline{A^c}$

[Notes 2/9 p.5: Prop 1 (iii), Prop 2 (ii)]

Cor: $\partial A = \partial(A^c)$:)

4. Let A be the interval $[0,1) \subseteq \mathbf{R}$ with the subspace topology.

(a) Explain why A is not compact.

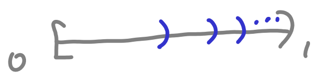
(b) Prove it directly by exhibiting an open cover of A that has no finite subcover.

a: $[0,1)$ is not closed in $\mathbb{R}$ ($\overline{[0,1)} = [0,1]$)

So by the Heine-Borel theorem $[0,1)$ is not compact.

b: (i) For $n > 1$ let $U_n = (-\infty, 1-\frac{1}{n})$ (open in $\mathbb{R}$)

Then $U_n \cap [0,1) = [0, 1-\frac{1}{n})$ are open in $[0,1)$.



(ii) If $x \in [0,1)$, $x<1$. By the Archimed. principle

$\exists\, n > \frac{1}{1-x}$, so $\frac{1}{n} < 1-x$, so $x < 1-\frac{1}{n}$, so $x \in [0, 1-\frac{1}{n})$

so $\{[0, 1-\frac{1}{n}) : n > 1\}$ is an open cover of $[0,1)$

(iii) Let $\{[0, 1-\frac{1}{n_k}) : k = 1, \dots m\}$ be a finite subcollection.

Then $\bigcup_{k=1} [0, 1-\frac{1}{n_k}) = [0, 1-\frac{1}{M})$, where $M = \max_{k=1}^{m} n_k$

Pf: $n_k \leq M \Rightarrow \frac{1}{n_k} \geq \frac{1}{M} \Rightarrow 1-\frac{1}{n_k} \leq 1-\frac{1}{M}$, so $[0, 1-\frac{1}{n_k}) \subseteq [0, 1-\frac{1}{M})$

Since $M > 0$, $1-\frac{1}{M} < 1$ so $[0, 1-\frac{1}{M}) \subsetneq [0,1)$

so our finite subcollection is not a cover :(