Midterm / 2022.10.6 / MAT 4233.001 / Modern Abstract Algebra

1. Determine for which natural numbers we have $n! > 2^n$ and prove it by induction.

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create_list([n,n!,2^n],n,1,7);
[[1,1,2],[2,2,4],[3,6,8],[4,24,16],[5,120,32],[6,720,64],[7,5040,128]]
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$n! > 2^n$ for $n \geq 4$

Pf. by induction on n:

Basis: $n=4$: $n! = 24 > 16 = 2^4$ ☺

Inductive step: Let $n > 4$, assume $\forall k < n$ $k! > 2^k$

In particular, since $n-1 < n$ assume $(n-1)! > 2^{n-1}$!

Then $n! = n(n-1)! > n\,2^{n-1} > 2 \cdot 2^{n-1} = 2^n$ ☺

2. Suppose $\alpha, \beta \in \Sigma_n$ are permutations of $\{1, 2, 3, 4, 5\}$ given by

x	1	2	3	4	5
$\alpha(x)$	3	2	4	1	5

x	1	2	3	4	5
$\beta(x)$	4	5	1	3	2

Find $\underbrace{(\alpha\beta)^{-1}}_{\beta^{-1}\alpha^{-1}}$ and $\alpha^{-1}\beta^{-1}$

x	1	2	3	4	5
$\alpha^{-1}(x)$	4	2	1	3	5
$\beta^{-1}(x)$	3	5	4	1	2
$\beta^{-1}(\alpha^{-1}(x))$	1	5	3	4	2
$\alpha^{-1}(\beta^{-1}(x))$	1	5	3	4	2

Cycle notation:

$\alpha = (1\ 3\ 4)$

$\beta = (1\ 4\ 3)(2\ 5)$

$\alpha^{-1} = (1\ 4\ 3)$

$\beta^{-1} = (1\ 3\ 4)(2\ 5)$

$\beta^{-1}\alpha^{-1} = (2\ 5)$

$\alpha^{-1}\beta^{-1} = (2\ 5)$

3. Suppose G is a finite group and $x \in G$. Prove:

(a) x has finite order.

(b) $x^n = e$ if and only if the order of x divides n.

(a) Since G is finite, by the pigeonhole principle

not all $x^i, i \geq 1$ are distinct, so $\exists\ i > j \quad x^i = x^j$

Then $x^j = x^i = x^{i-j} x^j$, so $x^{i-j} = e$

By the well-ordering principle $\{k > 0 : x^k = e\}$ has a

minimum called the order of x (denoted $|x|$)

(b) Let $m = |x|$

If $m \mid n$, $\exists\ q \in \mathbb{Z} \quad n = mq$, so $x^n = x^{mq} = (x^m)^q = e^q = e$ ◡̈

Conversely, suppose $x^n = e$

By the division algorithm $\exists!\ q, r \in \mathbb{Z} \quad n = mq + r, \quad 0 \leq r < m$

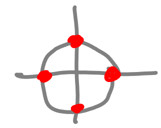
$r = n - mq \qquad x^r = x^{n-mq} = x^n \cdot (x^m)^{-q} = e$

If $r > 0$, contradiction since $r < m$

$\therefore\ r = 0$ ◡̈

4. Prove that the set of all complex fourth roots of unity $H = \{z \in \mathbf{C} \colon z^4 = 1\}$ is a cyclic subgroup of $\mathbf{C}^*$ of order 4

$z^4 = 1 \iff z^4 - 1 = 0 \iff (z^2-1)(z^2+1) = 0$



$\iff (z-1)(z+1)(z-i)(z+i) = 0$

Thus $H = \{1, i, -1, -i\}$

Cayley table:

$\cdot$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

$H < \mathbb{C}^*$

Pf: apply the subgroup test:

(i) $1 \in H$

(ii) closure: Cayley table

(iii) inverses: every row has a 1

Fancy proof: $H = \ker f$, where $f: \mathbb{C}^* \to \mathbb{C}^*$, $f(z) = z^4$

$[f(z_1 z_2) = (z_1 z_2)^4 = z_1^4 \cdot z_2^4 = f(z_1) \cdot f(z_2)]$

$H = \langle i \rangle = \{i^k : k \in \mathbb{Z}\} = \{i^0, i^1, i^2, i^3\} = \{1, i, -1, -i\}$

Also $H = \langle -i \rangle$:)

In fact $\varphi: \mathbb{Z}_4 \to H$, $\varphi(k) = i^k$ is an isomorphism:

Hom: $\varphi(k_1 + k_2) = i^{k_1+k_2} = i^{k_1} \cdot i^{k_2} = \varphi(k_1)\varphi(k_2)$

Surjective: by inspection

Since $|\mathbb{Z}_4| = |H|$, φ is also injective