1a Suppose $x \in G$, $x = m + n\sqrt{2} = m' + n'\sqrt{2}$.

Then $(m - m') + (n - n')\sqrt{2} = 0$

If $n - n' \neq 0$ $\quad \sqrt{2} = \dfrac{m' - m}{n - n'} \in \mathbb{Q}$ ☹

$\therefore n = n'$, so $m = m'$

1b (i) operation (closure) If $m + n\sqrt{2}, m' + n'\sqrt{2} \in G$,

$m + n\sqrt{2} + m' + n'\sqrt{2} = m + m' + (n + n')\sqrt{2} \in G$.

real addition is associative ☺

(ii) identity: $0 = 0 + 0\sqrt{2} \in G$

(iii) inverses: If $m + n\sqrt{2} \in G$, $m, n \in \mathbb{Z}$, so $-m - n\sqrt{2} \in G$

2a (i) reflexive $\forall x \in \mathbb{R}$ $x - x = 0 \in G$, so $x \sim x$

(ii) symmetric If $x - x' \in G$, $x' - x \in G$

(iii) transitive If $x - x', x' - x'' \in G$, $x - x' + x' - x'' = x - x'' \in G$

2b $[0] = G = \{0, 1, 1 + \sqrt{2}, 2 - 3\sqrt{2} \text{ etc.}\}$

$[\pi] = \pi + G = \{\pi, \pi + 1, \pi + 1 + \sqrt{2}, \pi + 2 - 3\sqrt{2} \text{ etc.}\}$

2c Suppose $x \sim x'$, $y \sim y'$. Then $x - x', y - y' \in G$

So $x + y - x' - y' \in G$, so $x + y \sim x' + y'$

3a $\forall x, x' \in G$ $\varphi_s(x)\varphi_s(x') = sxs^{-1}sx's^{-1} = sxx's^{-1} = \varphi_s(xx')$,

so $\varphi_s \in \mathrm{End}\, G$

φ_s has a compositional inverse $\varphi_{s^{-1}}$, so $\varphi_s \in \mathrm{Aut}\, G$

$\forall y \in G$ $\varphi_s(\varphi_{s^{-1}}(y)) = \varphi_s(s^{-1}ys) = ss^{-1}yss^{-1} = y$

similarly $\forall x \in G$ $\varphi_{s^{-1}}(\varphi_s(x)) = s^{-1}sxs^{-1}s = x$

[such automorphisms are called inner]

3b $\varphi_s(x) = x \iff sxs^{-1} = x \iff sx = xs$