Midterm 3

① $|G| = m$ $\gcd(m, n) = 1$

$\phi : G \to G$ $\phi(x) = nx$

1. ϕ is a hom: Let $x, y \in G$

$\phi(x+y) = n(x+y) = nx + ny = \phi(x) + \phi(y)$ ☺

2. Injective: Let $x \in \ker\phi$, then $\phi(x) = 0$,

i.e. $nx = 0$.

Then $|x|$ divides n.

By Lagrange's theorem $|x|$ divides m.

$\therefore$ $|x|$ divides $\gcd(m, n) = 1$

$\therefore$ $|x| = 1$ $\therefore$ $x = 0$

$\therefore$ $\ker\phi$ is trivial ☺

(2) $3 \cdot 5 = 15 \equiv 1 \mod 14$

$$\phi(1) = \phi(3 \cdot 5) = 5\,\phi(3)$$
$$= 5 \cdot [1, 5] = [5, 25] = [1, 4] \;\; \ddot\smile$$

Check: $\phi(3) = 3\,\phi(1) = 3[1, 4]$
$$= [3, 12] = [1, 5] \;\; \ddot\smile$$

(3) $N \triangleleft G$, $\left|\frac{G}{N}\right| = n$ Show $\forall x \in G$ $x^n \in N$

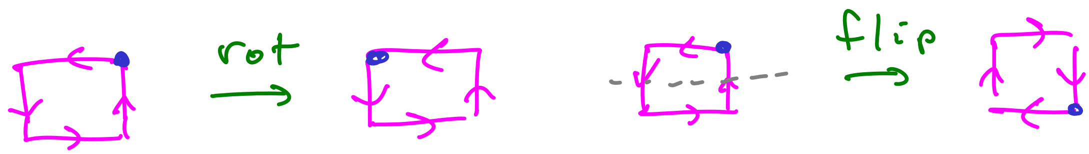
Recall: $\forall x \in G'$ $x^{|G'|} = e$ (cor. to Lagrange)

Let $x \in G$. Then $xN \in \frac{G}{N}$, so

$x^n N = (xN)^n = e_{G/N} = N$, so $x^n \in N$.

$(x^n = x^n e \in x^n N = N)$

④ D_n is partitioned into rotations (orientation preserving) and flips (orientation reversing).



$G < D_n \qquad \phi : G \longrightarrow \mathbb{Z}_2$

$$\phi(\tau) = \begin{cases} 0 & \text{if } \tau \text{ preserves orientation} \\ 1 & \text{otherwise} \end{cases}$$

Cases :

σ - rot , τ - rot $\qquad \phi(\sigma\tau) = 0$, $\phi(\sigma) + \phi(\tau) = 0 + 0 = 0$

σ - rot , τ - flip $\qquad \phi(\sigma\tau) = 1$, $\phi(\sigma) + \phi(\tau) = 0 + 1 = 1$

σ - flip , τ - rot $\qquad \phi(\sigma\tau) = 1$, $\phi(\sigma) + \phi(\tau) = 1 + 0 = 1$

σ - flip , τ - flip $\qquad \phi(\sigma\tau) = 0$, $\phi(\sigma) + \phi(\tau) = 1 + 1 = 0$

$\therefore \phi$ is a hom

If $G = D_n$, then $\ker \phi$ = rotation group

But kernels are automatically normal. ☺