1. What hypotheses on m and n are needed to ensure that $\mathbf{Z}_{mn} \cong \mathbf{Z}_m \oplus \mathbf{Z}_n$? Show by example that if the hypotheses are not satisfied, then the conclusion fails to hold. Explain why your example works.

If $\gcd(m,n)=1$, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$

(theorem 8.2)

$\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

↑ has no elements of order 4

2. Exhibit a nontrivial proper subgroup of the symmetric group S_n that is normal. Same for not normal. Prove your assertions.

$A_n \triangleleft S_n$

Suppose $\sigma \in S_n$ $\tau \in A_n$

$$\underbrace{\underbrace{\sigma^{-1}}_{\text{same parity}} \overset{\text{even}}{\tau} \underbrace{\sigma}_{\text{same parity}}}_{\text{even}} \in A_n$$ ☺

Alternate: define $\phi : S_n \to \mathbb{Z}_2$

by $\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$

By inspection ϕ is an onto hom.

(e.g. if σ is even, τ is odd

then $\sigma \cdot \tau$ is odd

$\phi(\sigma) + \phi(\tau) = 1 + 0 = 1 = \phi(\sigma\tau)$

etc.)

$A_n = \ker \phi$ $\therefore$ normal in S_n ☺

Assume $n \geq 3$ (S_1 is trivial and $S_2 \cong \mathbb{Z}_2$)

Let $H = \{ \sigma \in S_n : \sigma(1) = 1 \}$

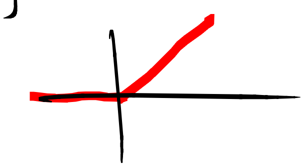
$(23) \in H$ $\qquad$ $(12)^{-1} = (12)$

$(12)(23)(12) = (13) \notin H$ $\quad \therefore H \ntrianglelefteq S_n$

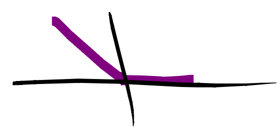
3. Let R be the ring of continuous functions $\mathbf{R} \to \mathbf{R}$ with the usual pointwise subtraction and multiplication. Which elements of R are units? Are there nonzero zero divisors in R? Let $A = \{f \in R \colon f(0) = 0\}$. Prove that A is an ideal of R. Is A a prime ideal? Maximal? Prove your assertions.

Units $= \{ f \in R : f(x) \neq 0 \ \forall x \in \mathbb{R} \}$

Zero-div? Let $f(x) = \begin{cases} x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$



$f(x) \cdot f(-x) \equiv 0$



Let phi be the evaluation (at 0) map from R -> **R**. In other words given f(x) in R, phi(f)=f(0). Show that phi is a ring hom. Then show that phi is onto and ker phi = A. Apply the 1st isomorphism theorem to deduce that R/A is isomorphic to **R**. Since R/A is a field, A is maximal.

Alternate: The pedestrian technique is slightly more tricky here, because we are dealing with just continuous functions, so don't have polynomial structure (coefficients, etc.) to rely upon. Pick an ideal A' strictly bigger than A and pick f in A'\A. Then f(0) neq 0. Since f may have other zeros besides x=0, consider instead for example h(x)=f(x)^2+|x|. Since f(x)^2 is in A' and |x| is in A, h(x) is in A', but it is clearly strictly positive everywhere, so has no zeros, so is a unit in R. Done.

4. Prove that $x^2 + 1$ is an irreducible polynomial in $\mathbf{R}[x]$. Prove that the factor ring $\mathbf{R}[x]/\langle x^2+1\rangle$ is a field.

If x^2+1 factors, it must factor into linears,

But those have zeros, whereas $x^2+1 > 0$

(Theorem 17.1)

Since x^2+1 is irreducible, $\langle x^2+1 \rangle$ is a maximal ideal.

(Theorem 17.5)

Therefore $\dfrac{\mathbb{R}[x]}{\langle x^2+1\rangle}$ is a field.

(Theorem 14.4)