Midterm 2 / 2008.4.24 / Modern abstract algebra / MAT 4233.001

Note Title 4/24/2008

1. Let $H = \{(), (12)(34), (13)(24), (14)(23)\}$. Prove that H is a subgroup of A_4. What is its index $[A_4 : H]$? Is H normal in A_4? Prove your assertion.

subgroup

Identity: $() \in H$ ☺

Inverses: Each element of H is its own inverse, e.g. $(12)(34)(12)(34) = ()$ and the rest are similar. ☺

Closure: It remains to check products of nontrivial pairs, e.g. $(12)(34)(13)(24) = (14)(23)$ and the rest are similar ☺

Index: $|A_4| = |S_4|/2 = 4!/2 = 12$

$|H| = 4$ $\therefore$ By Lagrange's theorem $[A_4 : H] = \frac{12}{4} = \boxed{3}$

Normality: Yes. The elements of A_4 not in H are the 3-cycles: $(1\,2\,3)$, $(1\,3\,2)$, $(1\,2\,4)$, $(1\,4\,2)$, $(1\,3\,4)$, $(1\,4\,3)$, $(2\,3\,4)$, $(2\,4\,3)$

Let's conjugate by (123) for example. Note: $(123)^{-1} = (132)$.

$(123)(12)(34)(132) = (14)(23)$

$(123)(13)(24)(132) = (12)(34)$

$(123)(14)(23)(132) = (13)(24)$

and the rest are similar ☺

2. Let A be the set of all polynomials in $\mathbf{Z}[x]$ such that the constant coefficient is divisible by 3. Prove that A is an ideal of $\mathbf{Z}[x]$. Is it maximal? Prove your assertion.

ideal

Identity: $3 \mid 0$ $\therefore 0 \in A$ ☺

Closure under differences: If $p = a_0 + a_1 x + \dots a_n x^n$, $q = b_0 + b_1 x + \dots + b_m x^m \in A$, then $3 \mid a_0$ and $3 \mid b_0$, so $3 \mid (a_0 - b_0)$.

$\therefore p - q = (a_0 - b_0) + (a_1 - b_1)x + \dots \in A$ ☺

Absorption: If $p = a_0 + a_1 x + \dots a_n x^n \in A$, $q = b_0 + b_1 x + \dots + b_m x^m \in \mathbb{Z}[x]$, then $3 \mid a_0$, so $3 \mid a_0 b_0$.

$\therefore pq = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + \dots \in A$ ☺

Maximal? Yes. Suppose B is an ideal of $\mathbb{Z}[x]$ with $A \subsetneq B$.

Then $\exists\, p = a_0 + a_1 x + \ldots + a_n x^n \in B \setminus A$. Since $p \notin A$, $3 \nmid a_0$,

so $\gcd(3, a_0) = 1$, so $\exists\, s, t \in \mathbb{Z}$ such that $3s + a_0 t = 1$.

Let $q = 3s + pt = 3s + (a_0 + a_1 x + \ldots + a_n x^n)t$

$= 3s + a_0 t + a_1 t x + \ldots + a_n t x^n = 1 + a_1 t x + \ldots + a_n t x^n$

Since $3s \in A$, $3s \in B$. Also $pt \in B$ (by absorption), so $q \in B$.

On the other hand $h = a_1 t x + \ldots a_n t x^n \in A \subset B$ (it has const. coeff. 0),

so $1 = q - h \in B$, so $B = \mathbb{Z}[x]$ ☺

3. Let R be the ring of continuous functions $\mathbf{R} \to \mathbf{R}$ with the usual pointwise operations. Is there a function in R that is neither a zero divisor nor a unit of R? Provide an explicit example or prove that no such example exists.

Let $f(x) = x$. Then, since $\frac{1}{x}$ is not continuous at 0, f is not a unit.

Suppose f is a zero divisor. Then $\exists\, g \in R$ with $fg = 0$.

$\forall x \neq 0$ $f(x) \neq 0$, so $g(x) = 0$.

By continuity $g(0) = \lim_{x \to 0} g(x) = 0$, so $g \equiv 0$ ☹

4. Suppose R is an integral domain. What is the largest and what is the smallest possible number of elements of R that are their own cubes? Explain.

If $x^3 = x$, then $x^3 - x = x(x-1)(x+1) = 0$

Since R is a domain, $x = 0$, $x = 1$, or $x = -1$.

If $1 \neq -1$ (e.g. $\mathbb{Z}_3$), we have 3 possibilities.

If $1 = -1$ (e.g. $\mathbb{Z}_2$), then only 2. ☺