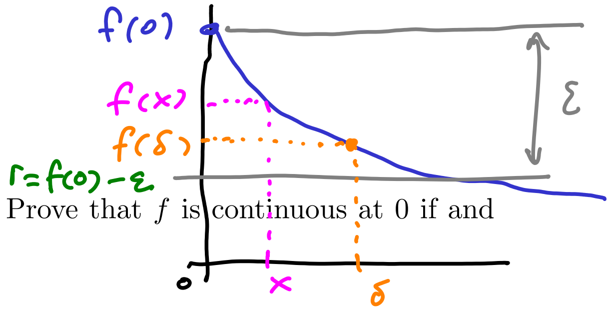

1. Suppose $f\colon [0,\infty) \to \mathbf{R}$ is a decreasing function. Prove that f is continuous at 0 if and only if $f(0) = \sup\{f(x)\colon x > 0\}$.

$\Rightarrow$ (i) $f(0)$ is an upper bound for $\{f(x) : x > 0\}$

Pf Since f is decreasing, $\forall x > 0$ $f(x) \le f(0)$:)

(ii) If $r < f(0)$, then r is not an upper bound for $\{f(x) : x > 0\}$.

Pf Let $\varepsilon = f(0) - r$. Then $\varepsilon > 0$, so since f is cont. at 0

$\exists \delta > 0$ $\underbrace{x \in [0,\infty) \ \& \ |x - 0| < \delta}_{0 \le x < \delta} \Rightarrow \underbrace{|f(x) - f(0)| < \varepsilon}_{\underbrace{f(0) - f(x) < f(0) - r}_{f(x) > r}}$

Pick any $x \in (0, \delta)$, e.g. $x = \frac{\delta}{2}$. Then $x > 0$ & $f(x) > r$:)

$\Leftarrow$ Given $\varepsilon > 0$, Let $r = f(0) - \varepsilon$.

Then $r < f(0) = \sup\{f(x) : x > 0\}$

so r is not an upper bound for $\{f(x) : x > 0\}$,

so $\exists \delta > 0$ $r < f(\delta)$

Suppose $x \in [0,\infty)$ & $|x - 0| < \delta$. Then $0 \le x < \delta$,

so since f is decr., $r = f(0) - \varepsilon < f(\delta) \le f(x) \le f(0)$.

$\therefore$ $0 \le f(0) - f(x) < \varepsilon$

so $|f(x) - f(0)| = f(0) - f(x) < \varepsilon$:)

2. Suppose $f(x) = x^3 \cos(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable at 0.

$$\frac{f(x)-f(0)}{x-0} = \frac{x^3\cos\left(\frac{1}{x}\right) - 0}{x-0} = \frac{x^3\cos\left(\frac{1}{x}\right)}{x} = x^2\cos\left(\frac{1}{x}\right) \to 0$$

as $x \to 0$

Pf Squeeze law: $0 \leq \left| x^2 \cos\left(\frac{1}{x}\right)\right| \leq x^2 \left|\cos\left(\frac{1}{x}\right)\right| \leq x^2$

0

:)

3. Prove that for $t > 1$ we have $\ln(t) < t - 1$.

Let $f(t) = \ln(t)$, then $f'(t) = \frac{1}{t}$

Let $t > 1$. By MVT $\exists x$ $1 < x < t$ s.t.

$$\underbrace{f(t) - f(1)}_{\ln t} = \underbrace{f'(x)}_{\frac{1}{x}}(t-1)$$

Since $x > 1 \Rightarrow \frac{1}{x} < 1$ $\quad \ln t < t - 1$ ☺