1. Show that if two continuous functions from reals to reals agree on rationals, they must be the same function.

a) If $c \in \mathbb{R}$ $\exists x_n \in \mathbb{Q}$ $x_n \to c$

Pf Since $\mathbb{Q}$ is dense in $\mathbb{R}$ $\forall n$ $\exists x_n \in \mathbb{Q}$ s.t.

$c < x_n < c + \frac{1}{n}$. Since $c + \frac{1}{n} \to c$

by the squeeze law $x_n \to c$:)

b) By the sequential criterion for continuity,

if $f, g : \mathbb{R} \to \mathbb{R}$ are cont.

$f(x_n) \to f(c)$ and $g(x_n) \to g(c)$

If f & g agree on $\mathbb{Q}$ $f(x_n) = g(x_n)$,

so by uniqueness of limit $f(c) = g(c)$

:)

2. Suppose $f : [0,1] \to [0,1]$ is continuous. Prove that f has a fixed point: $x \in [0,1]$ such that $f(x) = x$.

Let $g(x) = f(x) - x$

Then g is cont.

$g(0) = f(0) - 0 = f(0) \geq 0$

$g(1) = f(1) - 1 \leq 0$

If $f(0) = 0$ done, so WLOG assume $g(0) > 0$

if $f(1) = 1$ done, so WLOG assume $g(1) < 0$

By IVT $\exists z \in (0,1)$ s.t. $\underbrace{g(z)}_{f(z)-z} = 0$

so $f(z) = z$ ◡̈

3. Prove that the function $f(x) = \sqrt{x}$ is Lipschitz on the interval $[1, \infty)$. Why can we conclude that f is uniformly continuous on $[0, \infty)$?

a) For $x, y \in [1, \infty)$, $x, y \geq 1$, so

$\sqrt{x}, \sqrt{y} \geq 1$, so $\sqrt{x} + \sqrt{y} \geq 2$, so $\dfrac{1}{\sqrt{x}+\sqrt{y}} \leq \dfrac{1}{2}$

$$|\sqrt{y} - \sqrt{x}| = \frac{|y-x|}{\sqrt{y}+\sqrt{x}} \leq \frac{1}{2}|y-x| \quad ☺$$

b) Since f is Lipschitz on $[1, \infty)$,
f is unif. cont. on $[1, \infty)$

(Given $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{K} = 2\varepsilon$ etc.)

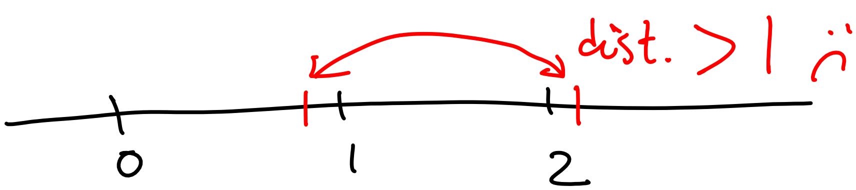
By the Uniform Continuity theorem,
f is unif. cont. on $[0, 2]$

Combine intervals: $[0,2] \cup [1,\infty) = [0,\infty)$

Given $\varepsilon > 0$

let $\delta = \min(\delta_1, \delta_2, 1)$ For $|x-y| < 1$, x, y are in one of the intervals

δ_1 from unif. cont. on $[0,2]$ $\quad$ δ_2 from unif. cont. on $[1,\infty)$



4. Give an example of a function $f : (0, 1) \to \mathbf{R}$ that is bounded, continuous, but not uniformly continuous. Explain.

Let $f : (0,1) \to \mathbb{R}$ be $f(x) = \cos\left(\frac{1}{x}\right)$

$\left|\cos\left(\frac{1}{x}\right)\right| \le 1$ so f is bounded.

Since $x \neq 0$ on $(0,1)$, $\frac{1}{x}$ is cont.

Also $\cos(x)$ is cont., so the composition

$\cos\left(\frac{1}{x}\right)$ is cont. on $(0,1)$

Let $x_n = \frac{1}{n\pi}$. Then $x_n \to 0$, so

(x_n) is a Cauchy seq.

Uniformly cont. functions carry Cauchy seq. to Cauchy seq., but

$f(x_n) = \cos(n\pi) = (-1)^n$ is not Cauchy

$\therefore$ f is not unif. cont. ☺