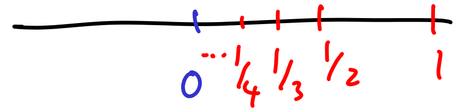
① a) $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$



Given $\delta > 0$, by the Archimedean property

$\exists n \in \mathbb{N}$ $n > \frac{1}{\delta}$, then $\frac{1}{n} \neq 0$

$\frac{1}{n} < \delta \iff n > \frac{1}{\delta}$

$\frac{1}{n} < \delta$

$\frac{1}{n} \in A$

$\therefore \frac{1}{n} \in A \cap V_\delta(0) \setminus \{0\}$ ☺

b) Def of limit: If $A \subseteq \mathbb{R}$, $g: A \to \mathbb{R}$, $c \in \mathbb{R}$, $L \in \mathbb{R}$,

we say $\lim_{x \to c} g(x) = L$ when

1. c is a cluster pt. of A
2. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\underset{x \in A}{0 < |x - c|} < \delta \Rightarrow |f(x) - L| < \varepsilon$

In our case: $c = 0$, $L = 0$, $g(x) = x \underbrace{f(x)}_{\uparrow \text{bdd}}$

1. Follows from (a)
2. Since f is bdd, $\exists M \in \mathbb{R}$, $M > 0$ s.t. $\forall x \in A$ $|f(x)| \leq M$

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$, then $\delta > 0$ and

if $x = \frac{1}{n} < \delta$, then

$\boxed{\text{Note: If } x \in A \ \exists n \ x = \tfrac{1}{n}}$

$$\left| \frac{1}{n} f\left(\frac{1}{n}\right) \right| = \frac{1}{n} \left| f\left(\frac{1}{n}\right) \right| \leq \frac{1}{n} M < \delta M = \varepsilon$$ ☺

Notice: $0 \leq |x f(x)| = |x| \, |f(x)| \leq |x| M$

(2) Let $\delta > 0$, by Achimedean property $\exists n > \frac{1}{2\pi\delta}$

Then $2\pi n > \frac{1}{\delta}$ $\qquad \underbrace{[2\pi n, 2\pi(n+1)]}_{\text{period of sine}} \subseteq (\frac{1}{\delta}, \infty) = V_\delta(\infty)$

$f(x) = \sin(x)$

$$f_*(V_\delta(\infty)) = \underbrace{[-1,1]}_{\text{size}=2} \not\subseteq (L-\tfrac{1}{2}, L+\tfrac{1}{2}) = V_{\underbrace{\frac{1}{2}}_{\text{size}=1}}(L) \text{ for any } L$$

Note: Any $0 < \varepsilon \leq 1$ works

☺

Alternative $\frac{\pi}{2} n \to \infty$ $\qquad \sin(\frac{\pi}{2} n) = 1, 0, -1, 0, 1, 0, \ldots$ diverges

By the sequential criterion $\lim_{x \to \infty} \sin(x)$ DNE ☺

(3) $$\frac{\sin x}{x} = \underbrace{\frac{1}{x}}_{\to 0} \underbrace{\sin x}_{\to ?} \quad \text{as } x \to \infty$$

$$\underbrace{t}_{\to 0} \underbrace{f(\tfrac{1}{t})}_{\text{bdd}} \text{ as } t \to 0 \qquad \text{By Prob. 1} \qquad \underbrace{t}_{\frac{1}{x}} \underbrace{f(\tfrac{1}{t})}_{\sin(x)} \to \boxed{0} \text{ as } t \to 0 \qquad x \to \infty$$ ☺

Sandwich: $0 \leq \left|\frac{\sin x}{x}\right| = \left|\frac{1}{x}\right| |\sin x| \leq \left|\frac{1}{x}\right|$

$\qquad\qquad \to 0 \qquad\qquad\qquad\qquad\qquad 0 \leftarrow$

☺

(4) Given $c \in \mathbb{R}$, $\delta > 0$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ & $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$

$\exists\, r \in \mathbb{Q},\ z \in \mathbb{R}\setminus\mathbb{Q}$ s.t. $r, z \in V_\delta(c)$

$$\left.\begin{array}{l} 1 = d(r) \in d_*(V_\delta(c)) \\ 0 = d(z) \in d_*(V_\delta(c)) \end{array}\right\} \text{ so}$$

$$\underbrace{\{0,1\} = d_*(V_\delta(c))}_{\text{diameter}=1} \not\subseteq \underbrace{V_{1/4}(L)}_{\text{diameter}=\frac{1}{2}} \text{ for } \underline{\underline{\text{any}}}\ L$$

(works for any $0 < \varepsilon \le \frac{1}{2}$)

Alt. If $c \in \mathbb{Q}$ pick a seq. $z_n \in \mathbb{R}\setminus\mathbb{Q}$ $z_n \to c$

then $f(z_n) = 0 \not\to f(c) = 1$

(density)

If $c \in \mathbb{R}\setminus\mathbb{Q}$ pick $r_n \in \mathbb{Q}$ $r_n \to c$

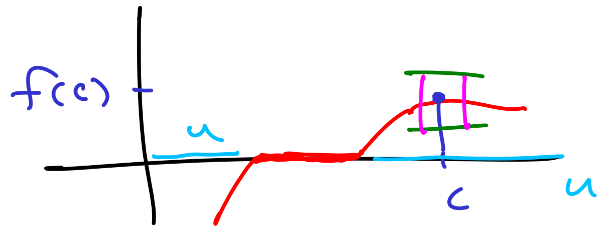
then $f(r_n) = 1 \not\to f(c) = 0$

b) Let $f(x) = d(x) - \frac{1}{2} = \begin{cases} 1/2 & \text{if } x \in \mathbb{Q} \\ -1/2 & \text{otherwise} \end{cases}$

If f is cont. @ c, $d(x) = f(x) + \frac{1}{2}$ would be cont. @ c (part (a))

$|f(x)| = 1/2$

(5)



Suppose f is cont. at c

$\exists \delta > 0 \quad f_*(V_\delta(c)) \subseteq V_{|f(c)|}(f(c))$

i.e. if $x \in V_\delta(c) \quad f(x) \in V_{|f(c)|}(f(c))$,

so $|f(c) - f(x)| < |f(c)|$

$$|f(c)| = |f(c) - f(x) + f(x)| \leq |f(c) - f(x)| + |f(x)| < |f(c)| + |f(x)|$$

$\therefore |f(x)| > 0 \quad \therefore f(x) \neq 0 \quad \therefore x \in U$

$\therefore V_\delta(c) \subseteq U$ ☺

Alternate method. Suppose $\forall \delta > 0 \quad V_\delta(c) \not\subseteq U$

In particular, $\forall n \in \mathbb{N} \quad V_{\frac{1}{n}}(c) \not\subseteq U$, so $\exists x_n \in V_{1/n}(c) \setminus U$

Then $x_n \to c$. Proof: Given $\varepsilon > 0 \quad \exists n > \frac{1}{\varepsilon}$.

Then $\forall k \geq n \quad k > \frac{1}{\varepsilon}$ so $\frac{1}{k} < \varepsilon$

$|c - x_k| < \frac{1}{k} < \varepsilon$ ☺

Since $x_n \notin U \quad f(x_n) = 0 \not\to f(c) \neq 0$ ☹