1. Suppose $A = \left\{ \dfrac{(-1)^n n}{n+1} : n \in \mathbf{N} \right\}$ and $f\colon A \to \mathbf{R}$ is a bounded function.

   (a) Find all cluster points of $A$ (with proof).

   (b) State the definition of limit and use it to prove that $(x-1)f(x) \to 0$ as $x \to 1$.

a) Cluster points of $A$ : $\boxed{1, -1}$

Pf As $n \to \infty$ $\quad \dfrac{n}{n+1} = \dfrac{1}{1 + 1/n} \to 1$

$\therefore$ Given $\varepsilon > 0$ $\exists m_\varepsilon$ $\forall k \geq m_\varepsilon$ $0 < \left| 1 - \dfrac{k}{k+1} \right| < \varepsilon$ $\quad \neq 1$ $\quad \left( \dfrac{k}{k+1} \in V_\varepsilon(1) \setminus \{1\} \right)$

If $k$ is even, $\dfrac{k}{k+1} \in A$, so $V_\varepsilon(1) \setminus \{1\} \cap A \neq \phi$

If $k$ is odd, $-\dfrac{k}{k+1} \in A$, so $\left| -1 + \dfrac{k}{k+1} \right| < \varepsilon$, so $V_\varepsilon(-1) \setminus \{-1\} \cap A \neq \phi$

$\quad = \left| 1 - \dfrac{k}{k+1} \right|$

$\therefore 1, -1$ are cluster pts. of $A$.

If $a \neq 1, -1$, $\varepsilon = \min\{ |a-1|, |a+1| \} > 0$ and

$V_\varepsilon(a) \cap V_\varepsilon(1) = V_\varepsilon(a) \cap V_\varepsilon(-1) = \phi$

$\therefore$ $V_\varepsilon(a) \setminus \{a\} \cap A \subseteq \left\{ \dfrac{(-1)^k}{k+1} : k \leq m_\varepsilon \right\} \leftarrow$ finite

$\therefore a$ is not a cluster pt. of $A$.

b) Given $g: A \to \mathbb{R}$, $c$ a cluster pt. of $A$, $\lim\limits_{x \to c} g(x) = L$ means

given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $x \in A$, $0 < |x - c| < \delta \Rightarrow |g(x) - L| < \varepsilon$

Since $f$ is bounded, $\exists M > 0$ $\forall x \in A$ $|f(x)| \leq M$

Given $\varepsilon > 0$ let $\delta = \dfrac{\varepsilon}{M}$. If $x \in A$ $0 < |x - 1| < \delta$,

$|(x-1) f(x)| = |x-1| \cdot |f(x)| \leq |x-1| M < \delta M = \dfrac{\varepsilon}{M} M = \varepsilon$ ☺

2. Prove that $\cos \frac{1}{x}$ fails to have a limit as $x \to 0$, while $\lim_{x \to 0} x \cos \frac{1}{x} = 0$.

Let $x_n = \dfrac{1}{n\pi}$, then $x_n \to 0$

But $\cos \frac{1}{x_n} = \cos(n\pi) = (-1)^n$, which diverges.

By the sequential criterion $\lim_{x \to 0} \cos \frac{1}{x}$ DNE

$0 \le |x \cos \frac{1}{x}| = |x| \, |\cos \frac{1}{x}| \le |x| \to 0$

By the squeeze lemma $|x \cos \frac{1}{x}| \to 0$ so $x \cos \frac{1}{x} \to 0$ as $x \to 0$

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and $A \subset \mathbf{R}$ is closed and bounded. Prove that the image $f_*(A)$ is closed in $\mathbf{R}$ and bounded.

If $f_*(A)$ is not bounded, $\forall n \in \mathbb{N}$ $\exists y_n \in f_*(A)$ $|y_n| > n$.

Since $|y_n| \to \infty$ as $n \to \infty$, any subsequence of $y_n$ diverges.

Since $y_n \in f_*(A)$, $\forall n \in \mathbb{N}$ $\exists x_n \in A$ $f(x_n) = y_n$

Since $A$ is bounded, by the Bolzano-Weierstrass theorem

$\exists$ subsequence $x_{n_k} \to$ some $x^* \in \mathbb{R}$.

Since $A$ is closed in $\mathbb{R}$, $x^* \in A$.

Since $f$ is continuous at $x^*$, by the sequential criterion

$f(x_{n_k}) = y_{n_k} \to f(x^*)$ $\therefore$ $f_*(A)$ is bounded.

Given $y_n \in f_*(A)$, $y_n \to y^*$, as before pick $x_n \in A$, $f(x_n) = y_n$

Since $A$ is bounded, $\exists$ subsequence $x_{n_k} \to$ some $x^* \in \mathbb{R}$

Since $A$ is closed, $x^* \in A$, so $y_{n_k} \to f(x^*)$

$\therefore$ $f(x^*) = y^*$, so $y^* \in f_*(A)$, so $f_*(A)$ is closed

4. Suppose $f : \mathbf{Q} \to \mathbf{R}$ can be extended continuously to $\mathbf{R}$. Prove that such an extension is unique.

Suppose $F, F' : \mathbb{R} \to \mathbb{R}$ is cont. and $F|_{\mathbb{Q}} = F'|_{\mathbb{Q}} = f$

($\forall x \in \mathbb{Q}$ $F(x) = F'(x) = f(x)$)

Let $a \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ seq. $x_n \in \mathbb{Q}$, $x_n \to a$.

Since $F, F'$ are cont., by the sequential criterion

$F(x_n) \to F(a)$, $F'(x_n) \to F'(a)$.

Since $F(x_n) = F'(x_n)$, by uniqueness of limits, $F(a) = F'(a)$. ☺

5. Suppose $f:\mathbf{R}\to\mathbf{R}$ is increasing and $c\in\mathbf{R}$. Prove that $f$ has a right limit at $c$.

Let $S = \{ f(x) : x > c \}$. Clearly $S \neq \emptyset$

Since $f$ is incr. $\forall x > c$ $\quad f(x) \geq f(c)$, so

$f(c)$ is a lower bound for $S$.

Since $\mathbb{R}$ is complete, $\exists\, y = \inf S$.

Given $\varepsilon > 0$ $\quad y + \varepsilon$ is not a lower bound for $S$, so

$\exists\, x^* > c \quad f(x^*) < y + \varepsilon$. Let $\delta = x^* - c > 0$.

If $c < x < c + \delta = x^*$, then $y \leq f(x) \leq f(x^*) < y + \varepsilon$

$\therefore\ y = \lim\limits_{x \to c^+} f(x)$

6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 \cos \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f$ is differentiable and $f'$ is not continuous at 0.

For $x \neq 0$ $\quad f'(x) = 2x \cos \frac{1}{x} + x^2 \left(-\sin \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}$

$f'(0) = \lim\limits_{x \to 0} \dfrac{x^2 \cos \frac{1}{x} - 0}{x - 0} = \lim\limits_{x \to 0} x \cos \frac{1}{x} = 0$ (see #2)

Suppose $\lim\limits_{x \to 0} f'(x) = f'(0)$. Then $\lim\limits_{x \to 0} \left(2x \cos \frac{1}{x} + \sin \frac{1}{x}\right) = 0$,

Since $\lim\limits_{x \to 0} 2x \cos \frac{1}{x} = 0$, $\lim\limits_{x \to 0} \sin \frac{1}{x} = 0$, but

Similarly to #2, $\lim\limits_{x \to 0} \sin \frac{1}{x}$ DNE

7. Find the limits at 0 and $\infty$ of $(1+\frac{1}{x})^x$ and prove your results.

$$\ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

If $x \to \infty$ we have $\frac{0}{0}$ and if $x \to 0$ we have $\frac{\infty}{\infty}$

l'Hôpital's rule: $\dfrac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \dfrac{x}{x+1} \begin{cases} \to 1 & \text{if } x \to \infty \\ \to 0 & \text{if } x \to 0 \end{cases}$

$\therefore \lim\left(1+\frac{1}{x}\right)^x = \begin{cases} e & \text{if } x \to \infty \\ 1 & \text{if } x \to 0 \end{cases}$

8. Let $f : (0,2) \to \mathbf{R}$ be defined by $f(x) = \ln x$. Let $p_n(x)$ denote the degree $n$ Taylor polynomial for $f$ at 1. Find $p_3$ and prove that $p_2(x) \le f(x) \le p_3(x)$ for all $x \in (0,2)$.

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$

$f''''(x) = -\frac{6}{x^4} \qquad f''''(1) = -6$

$$\therefore p_3(x) = \underbrace{x-1-\frac{1}{2}(x-1)^2}_{p_2(x)} + \frac{1}{3}(x-1)^3$$

By Taylor's theorem $\ln x = p_2(x) + \frac{1}{3}(y-1)^3$ for some $y$, $1 \le y \le x$.

Since $\frac{1}{3}(y-1)^3 \ge 0$, $\ln x \ge p_2(x)$

Also $\ln x = p_3(x) - \frac{1}{4}(y-1)^4$ for some $y$, $1 \le y \le x$

Since $\frac{1}{4}(y-1)^4 \ge 0$, $\ln x \le p_3(x)$