

1. Consider the differential equation $x^2y'' - xy' + (1-x)y = 0$.
 - (a) Find and classify all singularities.
 - (b) Construct and solve the indicial equation.
 - (c) Use the method of Frobenius to find two linearly independent solutions. For each of the two series involved compute the first three nontrivial terms.

Details can be found in Examples 8.6.4 and 8.7.2 in our textbook: R. K. Nagle, E. B. Saff, A. D. Snider, *Fundamentals of Differential Equations*, 6th ed., Addison Wesley, 2004

- (a) The only singularity is $x = 0$. It is regular.
 - (b) The indicial equation $r(r-1) - r + 1 = (r-1)^2$ has a double root $r = 1$.
 - (c) The method of Frobenius of §8.6 gives $y_1(x) = x + x^2 + \frac{1}{4}x^3 \dots$ and the method of § 8.7 gives a second linearly independent solution $y_2(x) = \ln(x)y_1(x) - 2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 \dots$
2. A thin rod with length $L = 1$ meter and diffusivity $\beta = 5$ has an initial temperature distribution $1 - \cos(\pi x)$ degrees Celsius for $0 \leq x \leq 1$. Assume that the ends of the rod are held at constant temperatures (what are they?).
 - (a) What is the temperature distribution for $t > 0$?
 - (b) What is the limit of your solution as $t \rightarrow \infty$, i.e. what is the steady state temperature distribution?

Evaluating the initial distribution at $x = 0, 1$ gives the boundary conditions $u(0, t) = 0$, $u(1, t) = 2$. Since the second derivative of a linear function is 0, the linear function satisfying the boundary conditions $2x$ is a time independent solution of the heat equation $u_t = \beta u_{xx}$.

By linearity of the heat equation $y(x, t) = u(x, t) - 2x$ is another solution. In addition, y satisfies the nicer boundary conditions $y(0, t) = y(1, t) = 0$. From the separation of variables (10.5.4) with $L = 1$

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(n\pi/L)^2 t} = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-5(n\pi)^2 t}$$

where c_n can be found by expanding $y(x, 0) = 1 - \cos(\pi x) - 2x$ in a sine Fourier series:

$$c_n = 2 \int_0^1 [1 - \cos(\pi x) - 2x] \sin(n\pi x) dx = -2 \frac{(-1)^n + 1}{n(n^2 - 1)\pi} = \begin{cases} -\frac{4}{n(n^2 - 1)\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Therefore,

$$\begin{aligned} u(x, t) &= 2x - \sum_{n=2,4,6,\dots} \frac{4}{n(n^2 - 1)\pi} \sin(n\pi x) e^{-5n^2\pi^2 t} \\ &= 2x - \frac{2}{3\pi} \sin(2\pi x) e^{-20\pi^2 t} - \frac{1}{15\pi} \sin(4\pi x) e^{-80\pi^2 t} - \frac{2}{105\pi} \sin(6\pi x) e^{-180\pi^2 t} \dots \end{aligned}$$

As $t \rightarrow \infty$ the solution approaches the steady state $u(x, \infty) = 2x$.