1. Use the definition to compute the Laplace transform of $t\,e^{-2t}\,u(t-3)$.
   For which s does the transform converge?

$$\int_0^\infty e^{-st}\,t\,e^{-2t}\,u(t-3)\,dt = \int_3^\infty t\,e^{-(s+2)t}\,dt = \left[-\frac{t\,e^{-(s+2)t}}{s+2}\right]_3^\infty + \int_3^\infty \frac{e^{-(s+2)t}}{s+2}\,dt$$

$$= \left[-\frac{te^{-(s+2)t}}{s+2} - \frac{e^{-(s+2)t}}{(s+2)^2}\right]_3^\infty = \left[-\frac{e^{-(s+2)t}}{(s+2)^2}[t(s+2)-1]\right]_{t=3}^{t=\infty} \to \frac{e^{-3s-6}(3s+5)}{(s+2)^2} \Leftrightarrow s > -2$$

2. Find the inverse Laplace transform of $\ln(s-4)$.

   Let $F = \ln(s-4)$. Since $\mathscr{L}[t^n f] = (-1)^n \dfrac{d^n F}{ds^n}$, we have

   $\mathscr{L}[tf] = -\dfrac{dF}{ds} = -\dfrac{1}{s-4} = \mathscr{L}[-e^{4t}]$, so $tf = -e^{4t}$, so $f = -\dfrac{e^{4t}}{t}$

3. Use the method of Laplace transforms to solve the initial value problem

$$x'' + x = u(t-3), \qquad x(0) = 1, \quad x'(0) = 2$$

   Take $\mathscr{L}$: $s^2X - s - 2 + X = -\dfrac{e^{-3s}}{s}$, so $(s^2+1)X = \dfrac{e^{-3s}}{s} + s + 2$. Solve for X:

$$X = \frac{e^{-3s}}{s(s^2+1)} + \frac{s+2}{s^2+1} = e^{-3s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

   Thus, $x = [1-\cos(t-3)]\,u(t-3) + \cos(t) + 2\sin(t)$

4. Find the Taylor series about $t = 0$ of $t^5(4+t^2)^{-1}$. Use the summation notation, but also write out the first three nonzero terms. What is the radius of convergence? Explain.

   Let $x = -t^2/4$. Then $t^2 = -4x$, so

$$\frac{t^5}{4+t^2} = \frac{t^5}{4-4x} = \frac{t^5}{4}\cdot\frac{1}{1-x} = \frac{t^5}{4}\sum_{k=0}^\infty x^k = \frac{t^5}{4}\sum_{k=0}^\infty \left(-\frac{t^2}{4}\right)^k = \sum_{k=0}^\infty \frac{(-1)^k}{4^{k+1}} t^{2k+5} = \frac{1}{4}t^5 - \frac{1}{16}t^7 + \frac{1}{64}t^9 + \ldots$$

   The nearest singularities to the origin are $2i$ and $-2i$, so the radius of convergence is 2.

   Alternately you can use the ratio test: $\left|\dfrac{(-1)^{k+1}t^{2k+7}}{4^{k+2}}\cdot\dfrac{4^{k+1}}{(-1)^k t^{2k+5}}\right| = \dfrac{|t|^2}{4} < 1 \Leftrightarrow |t| < 2$

5. Find the first three nonzero terms of the power series solution about $t = 0$ to the initial value problem

$$(t+1)x'' - x = 0, \qquad x(0) = 0, \quad x'(0) = 2$$

   Let $x = \sum_{k=0}^\infty a_k t^k$, where $a_0 = 0$, $a_1 = 2$. Then $x'' = \sum_{k=0}^\infty a_{k+2}(k+2)(k+1)t^k$,

   so $tx'' = \sum_{k=0}^\infty a_{k+2}(k+2)(k+1)t^{k+1} = \sum_{k=1}^\infty a_{k+1}(k+1)kt^k = \sum_{k=0}^\infty a_{k+1}(k+1)kt^k$

   Plug into $tx'' + x'' - x = 0$, collect the coefficients of x^k to obtain the recurrence relation $a_{k+1}(k+1)k + a_{k+2}(k+2)(k+1) - a_k = 0$, and solve:

$$a_{k+2} = \frac{a_k - a_{k+1}(k+1)k}{(k+2)(k+1)}, \text{ i.e. } a_k = \frac{a_{k-2} - a_{k-1}(k-1)(k-2)}{k(k-1)}$$

   Choosing $k = 2, 3, \ldots$ we can obtain further coefficients: $a_2 = \dfrac{a_0 - a_1\cdot 1\cdot 0}{2\cdot 1} = 0$,

   $a_3 = \dfrac{a_1 - a_2\cdot 2\cdot 1}{3\cdot 2} = \dfrac{1}{3}$, $a_4 = \dfrac{a_2 - a_3\cdot 3\cdot 2}{4\cdot 3} = -\dfrac{1}{6}$, so $x = 2t + \dfrac{1}{3}t^3 - \dfrac{1}{6}t^4\ldots$