Midterm 2 / 2009.12.2 / Calculus for Applications / MAT 3243.001

Note Title 12/4/2009

1. Use the method of Lagrange multipliers to find the global maximum and minimum of the scalar field $f(x,y) = 2x^2 + 3y^2$ on the unit disc.

$\operatorname{grad} f = [4x, 6y] = 0 \Rightarrow x = y = 0 \quad \therefore$ critical pt. $[0,0]$ ($f=0$, min)

Boundary: $g = x^2 + y^2 - 1 = 0$, $\operatorname{grad} g = [2x, 2y]$

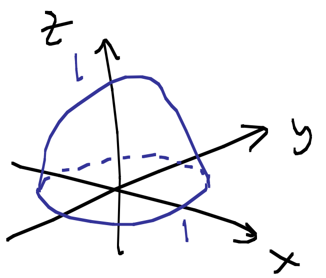
$\operatorname{grad} f = \lambda \operatorname{grad} g \Rightarrow 4x = \lambda 2x, \quad 6y = \lambda 2y$

$\therefore \quad x = 0, \lambda = 3$ or $y = 0, \lambda = 2$

$\therefore$ Boundary critical pts: $\underbrace{[0,1], [0,-1]}_{f=3}, \underbrace{[1,0], [-1,0]}_{f=2}$

max

2. Compute the volume of the solid enclosed by the surfaces $z = \underbrace{1 - x^2 - y^2}_{1-r^2}$ and $z = 0$.



$$\int_{-\pi}^{\pi}\int_0^1\int_0^{1-r^2} r \, dz\, dr\, d\theta = \int_0^1 rz\Big|_0^{1-r^2} dr \cdot \int_{-\pi}^{\pi} d\theta$$

$$= \int_0^1 (r - r^3)\, dr \cdot 2\pi = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] = \boxed{\frac{\pi}{2}}$$

3. Find the scalar potential for the vector field $F = [3x^2, z^2/y, 2z\ln y]$ or show that such a potential doesn't exist.

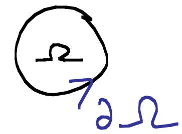
$V_x = 3x^2 \Rightarrow V = x^3 + f(y,z)$

$V_y = \frac{z^2}{y} \Rightarrow f_y = \frac{z^2}{y} \Rightarrow f = z^2 \ln y + g(z)$

$V_z = 2z \ln y \Rightarrow g'(z) = 0 \Rightarrow g = \text{const}$

$\therefore \boxed{V = x^3 + z^2 \ln y + C}$

4. Integrate $\omega = y\,dx - x\,dy$ around the unit circle counterclockwise. Compute the same integral using Green's theorem.



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} d\theta$$

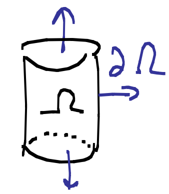
$$\omega = -(\sin\theta)^2 d\theta - (\cos\theta)^2 d\theta = -d\theta$$

$$\therefore \int_{\partial\Omega} \omega = -\int_{-\pi}^{\pi} d\theta = \boxed{-2\pi}$$

$$d\omega = dy\,dx - dx\,dy = -2\,dx\,dy$$

$$\int_{\Omega} d\omega = -2\int_{\Omega} dx\,dy = -2\,A(\Omega) = \boxed{-2\pi}$$

5. Compute the flux of $F = [2x, 3y, 0]$ through the surface $x^2 + y^2 = 1, -1 \le z \le 1$ oriented with the normal away from the z axis both directly and also using the divergence theorem.



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ z \end{bmatrix}, \quad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} -\sin\theta\, d\theta \\ \cos\theta\, d\theta \\ dz \end{bmatrix}, \quad d\bar{S} = \begin{bmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} d\theta\, dz$$

$$\int_{\text{side}} \bar{F} \cdot d\bar{S} = \int_{-1}^{1} \int_{-\pi}^{\pi} [2\cos\theta, 3\sin\theta, 0] \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} d\theta\, dz$$

$$= \int_{-1}^{1} dz \int_{-\pi}^{\pi} [2\underbrace{(\cos\theta)^2}_{\frac{1}{2} \text{ on average}} + 3\underbrace{(\sin\theta)^2}] d\theta = 2\left[1 + \tfrac{3}{2}\right] 2\pi = \boxed{10\pi}$$

$$\int_{\Omega} \operatorname{div} F\, dV = \int_{\partial\Omega} \bar{F} \cdot d\bar{S} = \int_{\text{side}} \bar{F} \cdot d\bar{S} + \underbrace{\int_{\text{top}} \bar{F} \cdot d\bar{S} + \int_{\text{bottom}} \bar{F} \cdot d\bar{S}}_{\text{Both 0, since } F \text{ is tangent to the surface}}$$

$$\| $$

$$5\int_{\Omega} dV = 5\,\mathrm{Vol}(\Omega)$$

$$= 5 \cdot \pi \cdot 2 = \boxed{10\pi}$$

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```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected

#1

```
> f:=2*x^2+3*y^2;
```

$$f := 2\,x^2 + 3\,y^2$$

```
> g:=x^2+y^2-1;
```

$$g := x^2 + y^2 - 1$$

```
> grad(f,[x,y]); solve(convert(%,set),{x,y}): cp:=subs(%,[x,y]);
```

$$[4\,x, 6\,y]$$

$$cp := [0, 0]$$

```
> L:=f-lambda*g;
```

$$L := 2\,x^2 + 3\,y^2 - \lambda\,(x^2 + y^2 - 1)$$

```
> grad(L,[x,y,lambda]); solve(convert(%,set),{x,y,lambda});
  cp,seq(subs(%[i],[x,y]),i=1..4);
  map(xx->subs({x=xx[1],y=xx[2]},f),[%]);
```

$$[4\,x - 2\,\lambda\,x, 6\,y - 2\,\lambda\,y, -x^2 - y^2 + 1]$$

$$\{x = 0, \lambda = 3, y = 1\}, \{x = 0, \lambda = 3, y = -1\}, \{y = 0, \lambda = 2, x = 1\}, \{y = 0, \lambda = 2, x = -1\}$$

$$[0, 0], [0, 1], [0, -1], [1, 0], [-1, 0]$$

$$[0, 3, 3, 2, 2]$$

#2

```
> int(int(int(r,z=0..1-r^2),r=0..1),theta=-Pi..Pi);
```

$$\frac{\pi}{2}$$

#3

```
> grad(x^3+z^2*ln(y),[x,y,z]);
```

$$\left[3\,x^2, \frac{z^2}{y}, 2\,z\ln(y)\right]$$

#4

```
> omega:=y*dx-x*dy;
```

$$\omega := y\,dx - x\,dy$$

```
> X:=[cos(theta),sin(theta)];
```

$$X := [\cos(\theta), \sin(\theta)]$$

```
> dX:=diff(X,theta);
```

$$dX := [-\sin(\theta), \cos(\theta)]$$

```
> subs({x=X[1],y=X[2],dx=dX[1],dy=dX[2]},omega); simplify(%);
  int(%,theta=-Pi..Pi);
```

$$-\sin(\theta)^2 - \cos(\theta)^2$$

$$-1$$

$$-2\,\pi$$

```
> -2*Pi;
```

$$-2\,\pi$$

#5

```
> F:=[2*x,3*y,0];
```

$$F := [\,2\,x,\,3\,y,\,0\,]$$

```
> X:=[cos(theta),sin(theta),z];
```

$$X := [\,\cos(\theta),\,\sin(\theta),\,z\,]$$

```
> DX:=jacobian(X,[theta,z]);
```

$$DX := \begin{bmatrix} -\sin(\theta) & 0 \\ \cos(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

```
> N:=crossprod(col(DX,1),col(DX,2));
```

$$N := [\,\cos(\theta),\,\sin(\theta),\,0\,]$$

```
> subs({x=X[1],y=X[2],z=X[3]},F); sum(%[i]*N[i],i=1..3);
  int(int(%,theta=-Pi..Pi),z=-1..1);
```

$$[\,2\cos(\theta),\,3\sin(\theta),\,0\,]$$

$$2\cos(\theta)^2 + 3\sin(\theta)^2$$

$$10\,\pi$$

```
> X:=[r*cos(theta),r*sin(theta),-1];
```

$$X := [\,r\cos(\theta),\,r\sin(\theta),\,-1\,]$$

```
> DX:=jacobian(X,[r,theta]);
```

$$DX := \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \\ 0 & 0 \end{bmatrix}$$

```
> crossprod(col(DX,1),col(DX,2)); N:=simplify(%);
```

$$[\,0,\,0,\,\cos(\theta)^2\,r + \sin(\theta)^2\,r\,]$$

$$N := [\,0,\,0,\,r\,]$$

```
> subs({x=X[1],y=X[2],z=X[3]},F); sum(%[i]*N[i],i=1..3);
  int(int(%,r=0..1),theta=-Pi..Pi);
```

$$[\,2\,r\cos(\theta),\,3\,r\sin(\theta),\,0\,]$$

$$0$$

$$0$$

```
> X:=[r*cos(theta),r*sin(theta),1];
```

$$X := [\,r\cos(\theta),\,r\sin(\theta),\,1\,]$$

```
> DX:=jacobian(X,[r,theta]);
```

$$DX := \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \\ 0 & 0 \end{bmatrix}$$

```
> crossprod(col(DX,1),col(DX,2)); N:=simplify(%);
```

$$[0, 0, \cos(\theta)^2 r + \sin(\theta)^2 r]$$

$$N := [0, 0, r]$$

```
> subs({x=X[1],y=X[2],z=X[3]},F); sum(%[i]*N[i],i=1..3);
  int(int(%,r=0..1),theta=-Pi..Pi);
```

$$[2\,r\cos(\theta), 3\,r\sin(\theta), 0]$$

$$0$$

$$0$$

```
> diverge(F,[x,y,z])*Pi*2;
  int(int(int(diverge(F,[x,y,z])*r,r=0..1),theta=-Pi..Pi),z=-1..1);
```

$$10\,\pi$$

$$10\,\pi$$

```
>
```