Midterm 2 / 2008.12.2 / Calculus for Applications / MAT 3243.001

Note Title 12/2/2008

1. A solid is bounded by the coordinate planes and the plane $5x + 2y + z = 10$. Set up, but do not evaluate the iterated integral for the volume with the order of integration z, x, y.

```
> with(plots):
Warning, the name changecoords has been redefined
> 5*x+2*y+z=10;
  implicitplot3d(%,x=0..10,y=0..10,z=0..10,axes=normal,scaling=constrained,style=patchnogrid);
```

$$5\,x + 2\,y + z = 10$$

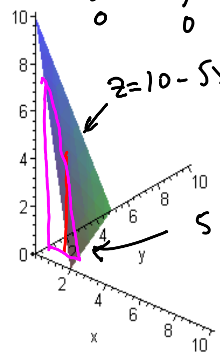
Intercepts:

$x = 2$

$y = 5$

$z = 10$

$$\int_0^5 \left[\int_0^{2-\frac{2}{5}y} \left[\int_0^{10-5x-2y} dz \right] dx \right] dy$$



$z = 10 - 5x - 2y$

$5x + 2y = 10$

$x = \frac{1}{5}(10 - 2y)$

$= 2 - \frac{2}{5}y$

2. Integrate $\omega = y\,dx + x\,dy$ along the segment of the curve $x^2 - y^5 = 0$ from $[-1, 1]$ to $[1, 1]$. Had we chosen a different path from $[-1, 1]$ to $[1, 1]$, would the integral remain the same? Explain.

Parametrize: $\bar{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t^5 \\ t^2 \end{bmatrix} \quad -1 \le t \le 1$

$d\bar{x} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 5t^4 \\ 2t \end{bmatrix} dt$, $\omega = t^2 \cdot 5t^4\,dt + t^5\,2t\,dt = 7t^6\,dt$

$\int \omega = \int_{-1}^{1} 7t^6\,dt = t^7 \Big|_{-1}^{1} = \boxed{2}$

$d\omega = dy\,dx + dx\,dy = 0$, so by FTC (or Green's theorem) the integral is path independent.

Alternate method: By inspection $\omega = d(xy)$

$\therefore \int \omega = \int d(xy) = xy \Big|_{[-1,1]}^{[1,1]} = 1 - 1 = \boxed{2}$

FTC

3. Find first a parametric formula and then an equation for the plane in $\mathbf{R}^3$ tangent to the surface $[st, s+t, e^{st}]$ at $[0, 1, 1]$.

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> X:=[s*t,s+t,exp(s*t)];
```

So to

$$X := [s\,t, s+t, e^{(s\,t)}]$$

```
> p:=[0,1]; subs({s=p[1],t=p[2]},X): P:=eval(%);
```

$$p := [0, 1]$$

$$P := [0, 1, 1]$$

← works! ☺

```
> jacobian(S,[s,t]); subs({s=p[1],t=p[2]},%): dX:=map(xx->eval(xx),%);
```

$$\begin{bmatrix} t & s \\ 1 & 1 \\ t\,e^{(s\,t)} & s\,e^{(s\,t)} \end{bmatrix}$$

← derivative

$$dX := \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

← eval.

linear approximation formula ↓

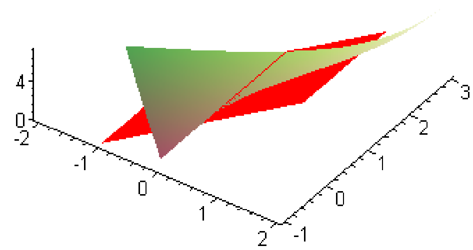
```
> P+dX&*[s-p[1],t-p[2]]; pl:=evalm(%);
```

$$[0, 1, 1] + (dX \,\&*\, [s, t-1])$$

$$pl := [s, s+t, 1+s]$$

← Parametric formula

```
> ps:=plot3d(S,s=-1..1,t=0..2,style=patchnogrid):
> pp:=plot3d(pl,s=-1..1,t=0..2,style=patchnogrid,color=red):
> display({ps,pp},axes=frame);
```




```
> Ts:=col(dX,1);
  Tt:=col(dX,2);
```

$$Ts := [1, 1, 1]$$

$$Tt := [0, 1, 0]$$

```
> N:=crossprod(Ts,Tt);
```

$$N := [-1, 0, 1]$$

```
> [x-P[1],y-P[2],z-P[3]]; ppl:=dotprod(%,N)=0;
```

$$[x, y-1, z-1]$$

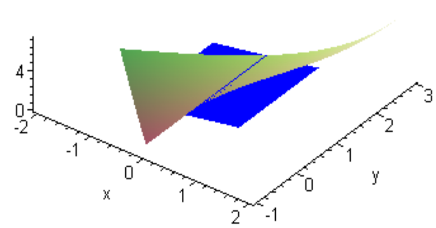
$$ppl := -x+z-1 = 0$$

← Equation

```
> solve(ppl,z); ppp:=plot3d(%,x=-1..1,y=0..2,style=patchnogrid,color=blue):
```

$$x+1$$

```
> display({ps,ppp},axes=frame);
```

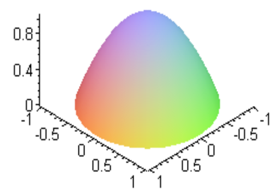


4. Parametrize the paraboloid $z = 1 - x^2 - y^2, z \geq 0$ oriented with the upward normal. Compute the flux of $\mathbf{F} = [x, y, z]$ through this surface. Would the flux of $\mathbf{F}$ through the unit disc differ? Explain.

```
> subs({x=r*cos(theta),y=r*sin(theta)},[x,y,1-x^2-y^2]): X:=simplify(%);
```

$$X := [r\cos(\theta), r\sin(\theta), 1 - r^2]$$

```
> plot3d(X,r=0..1,theta=-Pi..Pi,axes=frame,style=patchnogrid);
```




```
> dX:=jacobian(X,[r,theta]);
```

$$dX := \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \\ -2r & 0 \end{bmatrix}$$

$dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$dz = -2r\, dr$

$$d\vec{S} = \begin{bmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{bmatrix} = \begin{bmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ r \end{bmatrix} dr\, d\theta$$

```
> Tr:=col(dX,1);
  Tt:=col(dX,2);
```

$$Tr := [\cos(\theta), \sin(\theta), -2r]$$

$$Tt := [-r\sin(\theta), r\cos(\theta), 0]$$

```
> crossprod(Tr,Tt): N:=simplify(%);
```

$$N := [2r^2\cos(\theta), 2r^2\sin(\theta), r]$$

☺

$\mathbf{F} = X$ ☺

```
> sum(X[i]*N[i],i=1..3): simplify(%);
  int(int(%,r=0..1),theta=-Pi..Pi);
```

$$r(r^2+1)$$

$$\frac{3\pi}{2}$$

```
>
```

Since div F = 3, had we integrated through the unit disc,
by the divergence theorem
the answer would differ by 3 times the volume
of the solid paraboloid.

Note: Since on the unit disc $z = 0$ and N is vertical
(so has x & y components 0)
the answer would be 0 (by inspection ☺)

5. Find a scalar potential on the plane for the conservative vector field $[5x^2 - y, 2y^3 - x]$.

```
> (5/3)*x^3-x*y+y^4/2; jacobian([%],[x,y]);
```

$$\frac{5}{3}x^3 - xy + \frac{1}{2}y^4 \leftarrow \quad (+c)$$

$$[5x^2 - y, -x + 2y^3]$$