Midterm 1 / 2007.10.11 / Calculus for Applications / MAT 3243.001

Note Title

10/11/2007

1. Let $\mathbf{r} = [x, y, z]$ and $r = |\mathbf{r}|$. Express $\nabla \cdot (r^n \mathbf{r})$ in terms of r.

① The x component of $r^n \bar{r}$ is $r^n x$

$r = \sqrt{x^2+y^2+z^2}$ so $r_x = \frac{1}{\cancel{2}\sqrt{\dots}} \cdot \cancel{2}x = \frac{x}{r}$

$(r^n x)_x = (r^n)_x x + r^n = n r^{n-1} r_x x + r^n$

$= n r^{n-1} \frac{x}{r} \cdot x + r^n = n r^{n-2} x^2 + r^n$

Similarly $(r^n y)_y = n r^{n-2} y^2 + r^n$ & $(r^n z)_z = n r^{n-2} z^2 + r^n$

$\therefore \nabla \cdot (r^n \bar{r}) = n r^{n-2}(x^2+y^2+z^2) + 3r^n = n r^{n-2} r^2 + 3r^n$

$= n r^n + 3r^n = \boxed{(n+3) r^n}$

2. Let $\omega = y\,dx - x\,dy$ and $\eta = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$. Compute $d\omega$, $d\eta$ and $\omega \wedge \eta$.

② $\omega = y\,dx - x\,dy$, $\eta = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$d\omega = dy\,dx - dx\,dy = \boxed{-2\,dx\,dy}$

$d\eta = dx\,dy\,dz + dy\,dz\,dx + dz\,dx\,dy = \boxed{3\,dx\,dy\,dz}$

$\omega \wedge \eta = yx\,dx\,dy\,dz - xy\,dy\,dz\,dx = \boxed{0}$

3. Given a steady temperature distribution $f(x, y) = e^{xy}$, what is the rate of change of temperature as you start moving from the point $[1, 2]$ towards $[3, 3]$ with unit speed?

③ $f = e^{xy}$ Let $u = \begin{bmatrix}3\\3\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ $|u| = \sqrt{5}$

$\nabla f = [y e^{xy}, x e^{xy}]$ eval @ $\begin{bmatrix}1\\2\end{bmatrix}$: $[2e^2, e^2]$

$\nabla f \cdot \frac{u}{|u|} = \frac{4e^2 + e^2}{\sqrt{5}} = \boxed{\sqrt{5} e^2}$

4. Find the work done by the force $F = [y, x]$ in moving an object along the straight line segment from the origin to a point $[X, Y]$.

④ $F = [y, x]$ path : $\begin{bmatrix}0\\0\end{bmatrix}(1-t) + \begin{bmatrix}X\\Y\end{bmatrix} t = \begin{bmatrix}Xt\\Yt\end{bmatrix}$ $0 \le t \le 1$

$\begin{bmatrix}dx\\dy\end{bmatrix} = \begin{bmatrix}X\\Y\end{bmatrix} dt$

$\int \bar{F} \cdot d\bar{s} = \int [Yt, Xt] \cdot \begin{bmatrix}X\\Y\end{bmatrix} dt = \int (YXt + XYt)\,dt$

$= 2XY \int_0^1 t\,dt = 2XY \frac{t^2}{2}\Big|_0^1 = \boxed{XY}$

5. Find the flux of $F = [0, 0, x^2]$ through the unit disc oriented with the downward normal.

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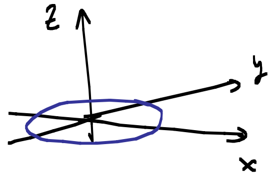

disc : $\begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix}$ $\quad 0 \le r \le 1$, $\quad -\pi < \theta \le \pi$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} dr \cdot \cos\theta - r\sin\theta \, d\theta \\ dr \sin\theta + r\cos\theta \, d\theta \\ 0 \end{bmatrix}$$

$$d\bar{S} = \begin{bmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} d\theta\, dr \qquad \bar{F} = [0, 0, r^2(\cos\theta)^2]$$

$$\int \bar{F} \cdot d\bar{S} = -\int_0^1 \left[\int_{-\pi}^{\pi} r^3 (\cos\theta)^2 \, d\theta \right] dr = -\int_{-\pi}^{\pi} (\cos\theta)^2 \, d\theta \cdot \int_0^1 r^3 \, dr$$

$$= -\int_{-\pi}^{\pi} \frac{1+\cos(2\theta)}{2} \, d\theta \cdot \left. \frac{r^4}{4} \right|_0^1 = -\left[\frac{1}{2}\theta + \frac{\sin(2\theta)}{4} \right]_{-\pi}^{\pi} \cdot \frac{1}{4}$$

$$= -\left[\frac{1}{2} \cdot 2\pi + 0 \right] \cdot \frac{1}{4} = \boxed{-\frac{\pi}{4}}$$