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Please show all work and justify your statements. Label sketches, draw conclusions using complete sentences including units, and box your final answers as appropriate.

1. Compute the following tangent approximations.
   (a) Find an equation for the plane tangent to the surface $2^x \ln z = y \sin(\pi y z)$ at $[2, 1, 1]$.
   (b) Find a parametric formula for the line tangent to $[\cos(\pi t), \sin(\pi t), 2t]$ at $[0, 1, 1]$.
2. Consider the vector field $F(x, y, z) = [x + yz, y + xz, z + xy]$.
   (a) Compute $DF$ and $\nabla \cdot F$.
   (b) Find a scalar potential for $F$. What conclusion can you make about $\nabla \times F$? Explain.
3. Suppose $s = f(u, v, w)$, where $u = y - z$, $v = z - x$, $w = x - y$. Express the partial derivatives of $s$ with respect to $x$, $y$, and $z$ in terms of the partial derivatives of $f$ with respect to $u$, $v$, and $w$.
4. For the scalar field $f(x, y) = 2x - 3y + \ln(xy)$ find all critical points and classify them using the Hessian criterion.

1) a) $2^x \ln z = y \sin(\pi y z)$

$$d(2^x)\ln z + 2^x d(\ln z) = dy \sin(\pi y z) + y\, d(\sin(\pi y z))$$

$$2^x \ln 2\, dx \ln z + 2^x \frac{1}{z} dz$$

$$= dy \sin(\pi y z) + y \cos(\pi y z)\, \pi [dy \cdot z + y \cdot dz]$$

$$2^x \ln 2 \ln z\, dx + [-\sin(\pi y z) - \pi y z \cos(\pi y z)]\, dy$$

$$+ \left[\frac{2^x}{z} - \pi y^2 \cos(\pi y z)\right] dz = 0$$

Eval @ $[2, 1, 1]$

$$0\, dx + \pi\, dy + [4 + \pi]\, dz = 0$$

Plane: $\boxed{\pi(y-1) + (4+\pi)(z-1) = 0}$

b) $f = \begin{bmatrix} \cos(\pi t) \\ \sin(\pi t) \\ 2t \end{bmatrix}$ $\quad f' = \begin{bmatrix} -\pi \sin(\pi t) \\ \pi \cos(\pi t) \\ 2 \end{bmatrix}$

$f(\frac{1}{2}) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad f'(\frac{1}{2}) = \begin{bmatrix} -\pi \\ 0 \\ 2 \end{bmatrix}$

Linear approx: $f(a) + f'(a)(t-a)$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\pi \\ 0 \\ 2 \end{bmatrix}(t - \tfrac{1}{2}) = \begin{bmatrix} -\pi t + \frac{\pi}{2} \\ 1 \\ 2t \end{bmatrix}$$

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2 a) $F = \begin{bmatrix} x + yz \\ y + zx \\ z + xy \end{bmatrix}$

$$DF = \begin{bmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{bmatrix}$$

$$\nabla \cdot F = \text{trace } DF = 1 + 1 + 1 = 3$$

b) Let $u = xyz + \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$

then $\nabla u = F$

$$\nabla \times F = \nabla \times (\nabla u) = 0,$$

Because curl (grad) is always 0

Another reason: DF is symmetric.

③ $\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$g = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} y - z \\ z - x \\ x - y \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Chain Rule:

$$\begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} f_u & f_v & f_w \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_w - f_v, & f_u - f_w, & f_v - f_u \end{bmatrix}$$

$$\therefore \quad f_x = f_w - f_v$$

$$f_y = f_u - f_w$$

$$f_z = f_v - f_u$$

(4) $f = 2x - 3y + \ln(xy)$

$$df = 2\,dx - 3\,dy + \frac{1}{xy}\left[dx \cdot y + x\,dy\right]$$

$$= \left[2 + \frac{1}{x}\right]dx + \left[-3 + \frac{1}{y}\right]dy$$

$df = 0 \Rightarrow x = -\frac{1}{2}, \ y = \frac{1}{3}$

However $\left[-\frac{1}{2}, \frac{1}{3}\right]$ is not in the domain of $f$

no points off if you didn't notice :)

So $f$ has no critical points

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Just for kicks, let's pretend that $f = 2x - 3y + \ln|xy|$

Then $\left[-\frac{1}{2}, \frac{1}{3}\right]$ is a critical point of $f$.

$$Hf = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix} \qquad \det Hf = \frac{1}{x^2 y^2}$$

@ $x = -\frac{1}{2}, y = \frac{1}{3}$ $\det Hf = 36 > 0$

So max or min

Since $f_{xx} = -\frac{1}{x^2}$ $f_{xx}\left(-\frac{1}{2}, \frac{1}{3}\right) = -4 < 0$

$\left[-\frac{1}{2}, \frac{1}{3}\right]$ is a local maximum of $f$.