1. Prove by induction that $n! \leq n^n$ for all natural numbers n.

   Basis of induction: $1! = 1$ and $1^1 = 1$, so $1! \leq 1^1$.

   Assume $(n-1)! \leq (n-1)^{n-1}$. Then $n! = n(n-1)! \leq n(n-1)^{n-1} \leq n \cdot n^{n-1} = n^n$. ☺

2. Use Euclid's algorithm to find the gcd and the Bézout coefficients for 58 and 44.

   $58 = 1 \cdot 44 + 14$, $44 = 3 \cdot 14 + 2$, $14 = 7 \cdot 2$, so $\gcd(58, 44) = 2$.

   Solve for remainders $2 = 44 - 3 \cdot 14$, $14 = 58 - 1 \cdot 44$ and back-substitute:
   $2 = 44 - 3(58 - 1 \cdot 44) = 4 \cdot 44 - 3 \cdot 58$ ☺

3. Suppose a, r, m are natural numbers with $a \equiv r \bmod m$. Prove that $\gcd(a, m) = \gcd(r, m)$.

   Since $a \equiv r \bmod m$, we have $a - r = mq$ for some $q \in \mathbf{Z}$, so $r = a - mq$.

   Since $\gcd(a, m)$ divides both a and m, it divides r.
   Since $\gcd(a, m)$ is a common divisor of r and m, it divides $\gcd(r, m)$.

   Conversely, since $\gcd(r, m)$ divides both r and m, it divides $a = mq + r$.
   Since $\gcd(r, m)$ is a common divisor of a and m, it divides $\gcd(a, m)$.

   Since the two gcd's are natural numbers dividing each other, they are equal. ☺

4. Find all solutions modulo 33 of the linear congruence $15x \equiv 21 \bmod 33$.

   Dividing by $\gcd(15, 33) = 3$ we obtain $5x \equiv 7 \bmod 11$.

   Since $5 \cdot (-2) \equiv 1 \bmod 11$, multiplying by -2 gives $x \equiv -14 \bmod 11 \equiv 8 \bmod 11$.

   (Euclid's algorithm: $11 = 2 \cdot 5 + 1$, so we get the Bézout relation $1 = 11 - 2 \cdot 5$)

   Thus, $x \equiv 8, 19, 30 \bmod 33$.

5. Prove that any nonzero element in a finite commutative ring with unity is either a unit or a zero divisor, but not both.

   Suppose R is a finite commutative ring with unity and $x \in R$. Assume $\mathbf{N} = \{0, 1, 2, ...\}$.

   Since R is finite, by the pigeonhole principle, some of the natural powers of x must agree.

   (Since $\mathbf{N}$ has more elements than R, no function $\mathbf{N} \to R$, particularly $i \mapsto x^i$, can be 1-1.)

   In other words, $x^i = x^j$ for some $i > j$. Then $x^i - x^j = 0$, so $x^j(x^{i-j} - 1) = 0$.

   Let $k = i - j$. Then $k > 0$ and $x^j(x^k - 1) = 0$. We may further assume j is minimal.

   (Let $S = \{m \in \mathbf{N} \colon x^m(x^k - 1) = 0\}$. Since $j \in S$, by the well ordering principle S has a minimum.)

   If $j = 0$, then $x^k - 1 = 0$, so $x \cdot x^{k-1} = x^k = 1$, so x is a unit.
   If $j > 0$, then $x^{j-1}(x^k - 1) \neq 0$, but $x \cdot x^{j-1}(x^k - 1) = 0$, so x is a zero divisor. ☺