1. Basis: $\sum_{k=1}^{1} k^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6}$. To prove the inductive step assume $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

   Then $\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n+1}{6}\left[n(2n+1) + 6(n+1)\right] = \frac{n+1}{6}\left[2n^2 + 7n + 6\right]$.

   Meanwhile $\frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6} = \frac{n+1}{6}\left[n+2\right]\left[2n+3\right] = \frac{n+1}{6}\left[2n^2 + 7n + 6\right]$.

2. Euclid's algorithm:

$31 = 2 \cdot 13 + 5$	$5 = 31 - 2 \cdot 13$
$13 = 2 \cdot 5 + 3$	$3 = 13 - 2 \cdot 5$
$5 = 3 + 2$	$2 = 5 - 3$
$3 = 2 + 1$	$1 = 3 - 2$

   Back substitution and combination of multiples gives a Bezout relation

   $1 = 3 - 2 = 3 - (5 - 3) = 2 \cdot 3 - 5 = 2 \cdot (13 - 2 \cdot 5) - 5 = 2 \cdot 13 - 5 \cdot 5 = 2 \cdot 13 - 5 \cdot (31 - 2 \cdot 13)$
   $= 12 \cdot 13 - 5 \cdot 31$,

   Therefore, $12 \cdot 13 \equiv 1 \bmod 31$.

   Multiplying $13x \equiv 2 \bmod 31$ by 12 gives $x \equiv 24 \bmod 31$.

3. Expand 45 in binary $45 = 32 + 8 + 4 + 1$ and compute repeated squares of 3 modulo 11:
   $3^2 = 9 = -2$, $3^4 = 4$, $3^8 = 16 = 5$, $3^{16} = 25 = 3$, $3^{32} = -2$.

   Therefore $3^{45} = 3^{32} \cdot 3^8 \cdot 3^4 \cdot 3 = (-2) \cdot 5 \cdot 4 \cdot 3 = -10 \cdot 12 = 1 \cdot 1 = 1$.

4. $(a, x) = b$ has a solution if and only if $b|a$.

   Proof: If $(a, x) = b$, then $b|a$. Conversely, if $b|a$, then $(a, b) = b$, so $x = b$ is a solution.

5. $(1 + i)(2 - 2i) = 2 + 2 = 4 = 0$, so $1 + i$ is a zero divisor.

   $(1 + 2i)(1 + 2i) = 1 - 4 + 4i = 1$, so $1 + 2i$ is a unit with order 2.