Midterm 2 / 2009.4.23 / MAT 3223.001 / Complex variables

Note Title 4/23/2009

1. Expand $(z-i)^{-1}+(z-2)^{-1}$ in a Laurent series centered at the origin and valid in the annulus $\{z : 1 < |z| < 2\}$.

$$\frac{1}{z-i} + \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{i}{z}} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} i^n \frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n} z^n$$

$$= \sum_{n=0}^{\infty} i^n \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n$$

$$= \sum_{n=1}^{\infty} i^{n-1}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n$$

$$\boxed{= \sum_{n=-\infty}^{-1} \frac{1}{i^{n+1}} z^n - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n}$$

2. Integrate $\cot z$ around the unit circle.

$\cot z = \frac{\cos z}{\sin z}$ has singularities $z = k\pi$, $k \in \mathbb{Z}$

only $z=0$ is inside the unit circle

$$\cot z = \frac{1-\frac{z^2}{2!}+\dots}{z-\frac{z^3}{3!}+\dots} = \frac{1}{z} + \dots$$

$\therefore$ Res $=1$, so $\int \cot z \, dz = \boxed{2\pi i}$

3. Use Rouché's theorem to determine the number of zeros, counted with multiplicity, of $z^3 \underbrace{-5z}_{\text{man}} + 1$ outside the unit disc.

$$f+g = \text{dog} = z^3+1$$
$$f = \text{man}+\text{dog} = z^3-5z+1$$
$$g = -\text{man} = 5z$$

If $|z|=1$

$$|f+g| = |z^3+1| \le |z|^3+1 = 2 < 5 = |5z| = |g| \le |f|+|g|$$

$\therefore$ Rouché applies, so f has the same # of zeros inside the unit disc as g, which is 1.

f is a cubic so has 3 zeros, so f has <u>2 zeros outside the unit disc.</u>

4. Find a fractional linear transformation that maps the unit disc to the right half-plane.

Pick 3 points on the unit circle counterclockwise: $-i, 1, i$ and map them to $0, 1, \infty$

Then $T(z) = \dfrac{z+i}{z-i} \cdot \dfrac{1-i}{1+i} = \dfrac{z+i}{z-i}\cdot(-i)$

takes the unit disc to the upper half plane.

Then rotate by $-\frac{\pi}{2}$, i.e. multiply by $e^{-i\frac{\pi}{2}} = -i$

Ans: $(-i)(-i)\cdot\dfrac{z+i}{z-i} = \boxed{-\dfrac{z+i}{z-i}}$

(not unique)