MAT 3223 Complex Var. Sp. 2002 Midterm 2

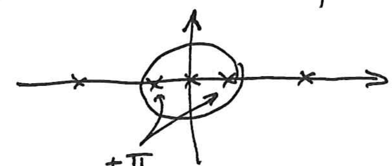
(1) Let $f(z) = z e^{4iz} + 2z + z e^{-4iz} = z\left(e^{4iz} + 2 + e^{-4iz}\right)$

$f(z) = 0 \Rightarrow \boxed{z = 0}$ or $e^{4iz} + 2 + e^{-4iz} = 0$

If $z \neq 0$ $e^{4iz} + 2 + e^{-4iz} = 0$, i.e. $\left(e^{4iz}\right)^2 + 2e^{4iz} + 1 = 0$

$\left(e^{4iz} + 1\right)^2 = 0$, $e^{4iz} = -1 = e^{i\pi}$, $4iz = i\pi + 2n\pi i, n \in \mathbb{Z}$

$$\boxed{z = \frac{\pi}{4} + \frac{n\pi}{2}, \ n \in \mathbb{Z}}$$



$\pm\frac{\pi}{4}$

$f'(z) = e^{4iz} + 2 + e^{-4iz} + z\left(4ie^{4iz} - 4ie^{-4iz}\right)$

$f'(0) = 1 + 2 + 1 \neq 0$ So $z = 0$ is a simple root.

$f'(\pm\frac{\pi}{4}) = 0 \pm \frac{\pi}{4}(-4i + 4i) = 0$ ∴

$f''(z) = 4ie^{4iz} - 4ie^{-4iz} + 4ie^{4iz} - 4ie^{-4iz} + z\left(-16e^{4iz} - 16e^{-4iz}\right)$

$= 8ie^{4iz} - 8ie^{-4iz} - 16z\left(e^{4iz} + e^{-4iz}\right)$

$f''(\pm\frac{\pi}{4}) = -8i + 8i \mp 4\pi(-1-1) = \pm 8\pi \neq 0$

So $\pm\frac{\pi}{4}$ have multiplicity 2.

(2) Let $p(z) = z^4 + z^2 - z - 5$

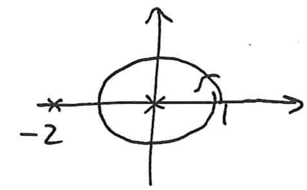
If $|z| = 2$, $|z^2 - z - 5| \leq |z^2| + |z| + |5| = 11 < 16 = |z^4|$

Rouché ⇒ in the disc $|z| < 2$ $p(z)$ has the same # of roots (with multiplicity) as z^4, i.e. 4

If $|z| = 1$, $|z^4 + z^2 - z| \leq 3 < 5 = |5|$

Rouché ⇒ in the disc $|z| < 1$ $p(z)$ has the same # of roots as 5, i.e. none.

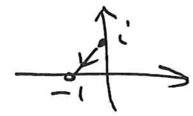
(3) a) $I = \int \frac{dz}{z^3 + 2z^2} = \int \frac{dz}{z^2(z+2)} = \int \frac{f(z)\,dz}{z^2}$



where $f(z) = \frac{1}{z+2}$, $f'(z) = -\frac{1}{(z+2)^2}$

$I = 2\pi i f'(0) = 2\pi i \left(-\frac{1}{4}\right) = \boxed{-\frac{\pi i}{2}}$

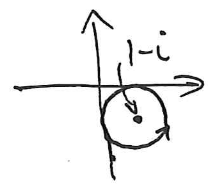
b) $z = i + t(-1-i)$, $dz = (-1-i)\,dt$



$\int \bar{z}\,dz = \int_0^1 [-i + t(-1+i)](-1-i)\,dt$

$= \int_0^1 (i - 1 + 2t)\,dt = i - 1 + 1 = \boxed{i}$

c) $z = 1 - i + e^{it}$, $dz = ie^{it}\,dt$



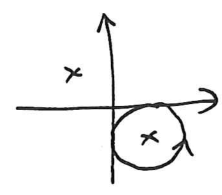
$\int \bar{z}\,dz = \int_0^{2\pi} (1 + i + e^{-it})\,ie^{it}\,dt$

Note: $|z - 1 + i| = 1 \Rightarrow z - 1 + i = e^{it}$

$= \int_0^{2\pi} [\underbrace{(i-1)e^{it}}_{\text{integrates to } 0} + i]\,dt = \boxed{2\pi i}$

d) $z^2 + i = 0$ $\quad z^2 = -i = e^{-i\frac{\pi}{2}}$ $\quad z = \pm e^{-i\frac{\pi}{4}} = \pm \frac{1-i}{\sqrt{2}}$

$I = \int \frac{z\,dz}{z^2 + i} = \int \frac{f(z)\,dz}{\left(z - \frac{1-i}{\sqrt{2}}\right)}$, where $f(z) = \frac{z}{z + \frac{1-i}{\sqrt{2}}}$



$I = 2\pi i f\left(\frac{1-i}{\sqrt{2}}\right) = 2\pi i \frac{\frac{1-i}{\sqrt{2}}}{\frac{1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}} = 2\pi i \cdot \frac{1}{2} = \boxed{\pi i}$

(4) $|I(r)| = \left|\int_\gamma \frac{1}{z^2+1}\,dz\right| \le \int_\gamma \frac{|dz|}{|z^2+1|} \overset{|z^2+1| \ge |z^2| - 1}{\le} \int_\gamma \frac{|dz|}{|z|^2 - 1} = \frac{1}{r^2-1}\int |dz|$

$= \frac{1}{r^2 - 1}\, 2\pi r/2 = \frac{\pi r}{r^2 - 1} \to 0$ as $r \to \infty$