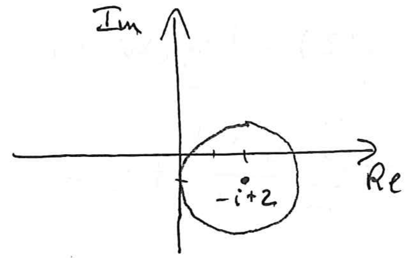
Midterm 1 MAT 3223 Complex Var. Spring 2002

1 a)

$$|z+i-2| = 2$$

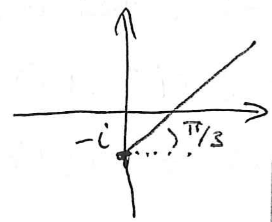
$$|z-(-i+2)| = 2$$



b)

$$\operatorname{Arg}(z+i) = \frac{\pi}{3}$$

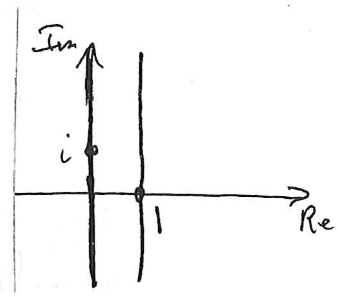
$$\operatorname{Arg}(z-(-i)) = \frac{\pi}{3}$$



c)

$$\operatorname{Re}(z-i) = 1$$

$\operatorname{Re}(z-i) = \operatorname{Re}(z) - \operatorname{Re}(i) = \operatorname{Re} z$



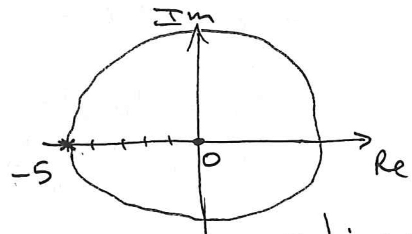
2 a)

$$\frac{1}{5+z} = \frac{1}{5}\,\frac{1}{1+\frac{z}{5}} = \frac{1}{5}\,\frac{1}{1-\left(-\frac{z}{5}\right)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{z}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}} z^n$$

$$= \frac{1}{5} - \frac{1}{25}z + \frac{1}{125}z^2 \ldots$$

singularities : -5

branch pts : none



radius = 5

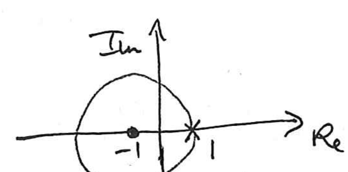
b) Let $w = z+1$, then $z = w-1$

$$\frac{1}{1-z} = \frac{1}{2-w} = \frac{1}{2}\,\frac{1}{1-\frac{w}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{w}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n$$

$$= \frac{1}{2} + \frac{1}{4}(z+1) + \frac{1}{8}(z+1)^2 \ldots$$

singularities : 1

branch pts : none



radius = 2

c) Let $w = z+1$

$$\log(z) = \log(w-1) = \int \frac{dw}{w-1} = \int -\sum_{n=0}^{\infty} w^n \, dw$$

$$= -\sum_{n=0}^{\infty} \frac{w^{n+1}}{n+1} + C = -\sum_{n=1}^{\infty} \frac{(z+1)^n}{n} + C$$

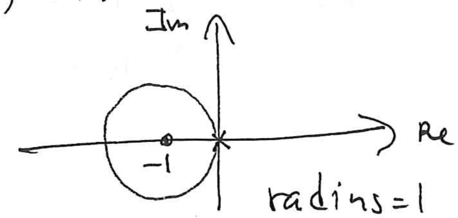
principal branch (k=0)

Let $z=-1$, then $\log(z) = \log(-1) = \log(e^{i\pi}) = i\pi + i2k\pi$, where $k \in \mathbb{Z}$

$\therefore \; C = i\pi(2k+1)$

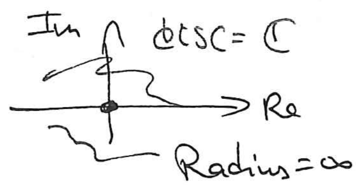
So $\log(z) = i\pi(2k+1) - \sum_{n=1}^{\infty} \frac{(z+1)^n}{n}$

$$= i\pi(2k+1) - (z+1) - \tfrac{1}{2}(z+1)^2 \ldots$$



Singularities : 0
Branch pts : 0

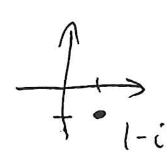
d) $$\frac{z^8}{e^{iz}} = z^8 e^{-iz} = z^8 \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} z^{n+8}$$



$$= z^8 - iz^9 - \frac{1}{2} z^{10} \ldots$$

Singularities: none
Branch pts: none

③ a) $f(z) = \log(z)$, $f'(z) = \frac{1}{z}$

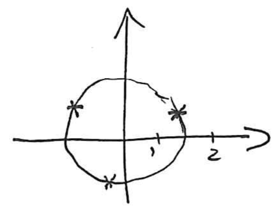


$$f'(1-i) = \frac{1}{1-i} = \frac{1}{\sqrt{2}\, e^{-i\pi/4}} = \frac{1}{\sqrt{2}} e^{i\pi/4}$$

Local Effect: dilation by $\frac{1}{\sqrt{2}}$, rotation by $\frac{\pi}{4}$

b) $f(z) = \frac{1}{z^2}$, $f'(z) = -\frac{2}{z^3}$

Solve $-\frac{2}{z^3} = \frac{1}{2} e^{i\frac{\pi}{2}} = \frac{i}{2}$



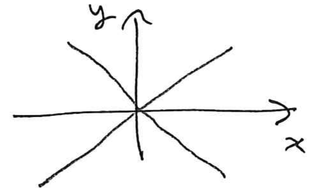
$$z^3 = -\frac{4}{i} = 4i = 4e^{i\pi/2}$$

$$z = \underbrace{4^{1/3}}_{\approx 1.59} e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)} \qquad k = 0, 1, 2$$

c) $f(z) = \frac{1}{2} e^{-i\pi/4} z + C$

4 a) $u = x^3$, $v = y^3$

$$J = \begin{bmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{bmatrix}$$

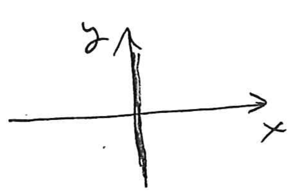


C/R $\Rightarrow$ $x^2 = y^2$ $\Rightarrow$ $x = \pm y$

$f' = 3x^2$

b) $u = x^2 + y^2$ $\quad v = -2xy$

$$J = \begin{bmatrix} 2x & 2y \\ -2y & -2x \end{bmatrix}$$



C/R $\Rightarrow$ $x = 0$

$f' = -2yi$