1. Suppose $\forall n\ x_n \in \mathbf{R}$ with $|x_n| < 1/n$. Prove that the sequence (x_n) is Cauchy directly from the definition. What is the limit of (x_n)?

Given $\varepsilon > 0$. By the Archimedean property $\exists k > \frac{2}{\varepsilon}$. If $n, m \geq k$, then $\frac{1}{n} \leq \frac{1}{k} < \frac{\varepsilon}{2}$ & $\frac{1}{m} \leq \frac{1}{k} < \frac{\varepsilon}{2}$

$$|x_n - x_m| \leq |x_n| + |x_m| < \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

☺

$$|x_n| < \frac{1}{n} \Rightarrow -\frac{1}{n} < x_n < \frac{1}{n}$$

0

$\therefore$ By the squeeze Law $x_n \to 0$ ☺

2. Suppose $\forall n\ a_n > 0$ and the series $\sum a_n$ converges.

   (a) Prove that $\sum a_n^2$ converges.

   (b) Show by example that $\sum \sqrt{a_n}$ need not converge.

a) Since $\sum a_n$ converges, by $n^{\underline{th}}$ term test

$a_n \to 0 \quad \therefore \exists k \ \forall n \geq k \quad \underline{0 < a_n < 1}$

Then $\forall n \geq k \quad \underline{a_n^2 < a_n}$

$\therefore$ By the comparison test $\sum a_n^2$ converges as well :)

b) $\sum \frac{1}{n^2}$ conv (by the p-test with $p = 2 > 1$)

But $\sum \frac{1}{n}$ div (harmonic series) :)

3. Use the definition of limit to prove that $x^2 + x + 1 \to 7$ as $x \to 2$.

Scratch work: $|x^2 + x + 1 - 7| = |x^2 + x - 6|$

$= |x-2||x+3|$

$$\begin{array}{r} x + 3 \\ x-2 \overline{\big)\, x^2 + x - 6} \\ \underline{x^2 - 2x} \\ 3x - 6 \end{array}$$

Pick a nbd of 2: $(2-1, 2+1)$
$= (1, 3)$

$x < 3 \Rightarrow x + 3 < 6$

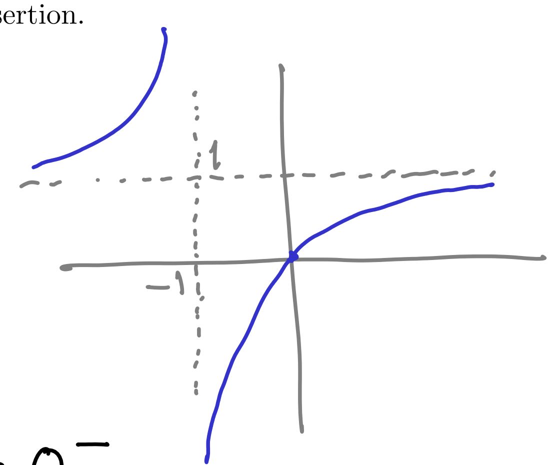
Given $\varepsilon > 0$, let $\delta = \min\{\underset{=}{1}, \frac{\varepsilon}{6}\}$. Then $\delta > 0$

and if $0 < |x-2| < \delta$, $|x^2 + x + 1 - 7| = |x^2 + x - 6|$

$= |x-2||x+3| < |x-2| \cdot 6 < \delta \cdot 6 \leq \frac{\varepsilon}{6} \cdot 6 = \varepsilon$ ☺

4. Find the limit of $\dfrac{x}{x+1}$ as $x \to -1^+$. Prove your assertion.

$$\lim_{x \to -1^+} \frac{x}{x+1} = -\infty$$



Pf

$$\lim_{x \to -1^+} \frac{x+1}{x} = \frac{-1^+ + 1}{-1^+} = \frac{0^+}{-1^+} = 0^-$$

$-1^+ > -1$

$-1^+ + 1 > 0$

$$\therefore \lim_{x \to -1^+} \frac{x}{x+1} = -\infty$$

:)