1. Suppose A, B are nonempty bounded subsets of $\mathbf{R}$. Let $A+B = \{a+b\colon a \in A, b \in B\}$. Prove that $\inf(A+B) = \inf A + \inf B$.

(i) $\inf A + \inf B$ is a lower bd for A:

Let $x \in A+B$, then $\exists\, a \in A, b \in B$ $x = a+b$

Since $a \in A$, $a \geq \inf A$. Sim: $b \geq \inf B$

$\therefore$ $x = a+b \geq \inf A + \inf B$. ☺

(ii) $\inf A + \inf B$ is the greatest lower bd for A:

Suppose $r > \inf A + \inf B$

Let $\delta = r - \inf A - \inf B$. Then $\delta > 0$, so $\frac{\delta}{2} > 0$.

$\inf A + \frac{\delta}{2}$ is not a lower bd for A,

so $\exists\, a \in A$ $a < \inf A + \frac{\delta}{2}$. Sim: $\exists\, b \in B$ $b < \inf B + \frac{\delta}{2}$

$$
\begin{aligned}
a+b &< \inf A + \tfrac{\delta}{2} + \inf B + \tfrac{\delta}{2} \\
&= \inf A + \inf B + \delta \\
&= \cancel{\inf A} + \cancel{\inf B} + r - \cancel{\inf A} - \cancel{\inf B} = r
\end{aligned}
$$

$\therefore$ r is not a lower bd for $A+B$. ☺

Alt. proof: If $a \in A$, $b \in B$, then $a \geq \inf A$, $b \geq \inf B$

So $a + b \geq \inf A + \inf B$

$\therefore$ $\inf(A+B) \geq \inf A + \inf B$.

$\forall\, a \in A, b \in B$ $\inf(A+B) \leq a+b$

$a \geq \inf(A+B) - b$

$\therefore$ $\inf A \geq \inf(A+B) - b$

$b \geq \inf(A+B) - \inf A$

$\inf B \geq \inf(A+B) - \inf A$

$\inf A + \inf B \geq \inf(A+B)$

$\therefore \inf A + \inf B = \inf(A+B)$

2. Prove that the sequence $\underbrace{(-1)^n \frac{n}{n+1}}_{x_n}$ diverges.

Pf 1 $\quad x_{2k} = \frac{2k}{2k+1} = \frac{2}{2+\frac{1}{k}} \to 1$ $\qquad \frac{1}{k} \to 0$

$\neq$

$x_{2k+1} = -\frac{2k+1}{2k+2} = -\frac{2+\frac{1}{k}}{2+\frac{2}{k}} \to -1$ $\qquad \frac{1}{k}, \frac{2}{k} \to 0$

Two subseq. w. different limits $\Rightarrow$ x_n is div.

Pf 2 $\quad$ Suppose $x_n \to x$, then $x_{n+1} \to x$

$\underbrace{|x_{n+1} - x_n|} \to |x - x| = 0$

$\left| (-1)^{n+1} \frac{n+1}{n+2} - (-1)^n \frac{n}{n+1} \right| = \left| (-1)^{n+1} \left[\frac{n+1}{n+2} + \frac{n}{n+1} \right] \right|$

$= \frac{n+1}{n+2} + \frac{n}{n+1} = \frac{n^2+2n+1+n^2+2n}{n^2+3n+2} = \frac{2n^2+4n+1}{n^2+3n+2}$

$= \frac{2+4/n+1/n^2}{1+3/n+2/n^2} \to 2$ $\qquad \to 0$

☹

Pf 3 $\quad$ Suppose $x_n \to x$. x cannot be both 1 & -1.
WLOG assume $x \neq 1$. Let $\delta = |x-1|/2$
(so $V_\delta(1) \cap V_\delta(x) = \emptyset$)

Since $x_n \to x$, $\exists k_1\ \forall n \geq k_1\ \ |x_n - x| < \delta$

Since $\frac{n}{n+1} = \frac{1}{1+1/n} \to 1$ $\quad (1/n \to 0)$ $\quad \exists k_2\ \forall n \geq k_2\ \ \left|\frac{n}{n+1} - 1\right| < \delta$

Pick even $n \geq \max\{k_1, k_2\}$

$|x-1| \leq |x-x_n| + |x_n - 1| < \delta + \delta = 2\delta = |x-1|$ ☹

3. Suppose $A \neq \varnothing$ and bounded below. Prove there is a sequence (a_n) in A such that $a_n \to \inf A$.

$\forall n$ $\quad \inf A + \frac{1}{n} > \inf A$, so $\inf A + \frac{1}{n}$ is not a lower bound for A

so $\exists\, a_n \in A$ $\quad a_n < \inf A + \frac{1}{n}$. Now apply squeeze law:

$$\inf A \leq a_n < \inf A + \frac{1}{n}$$

$$\inf A$$

$\therefore\ a_n \to \inf A$:)

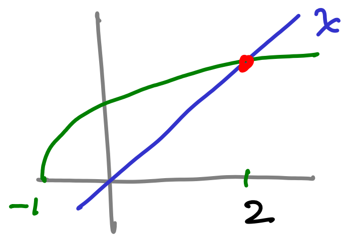
4. Suppose $x_1 = 1$ and $x_n = \sqrt{x_{n-1} + 2}$ for $n > 1$. Show that the sequence (x_n) is monotone increasing and bounded above, thus convergent. Find the limit.

Prelim. work: (i) Solve $x = \sqrt{x+2}$

$x^2 = x+2$

$x^2 - x - 2 = 0$

$x = 2, -1$



(ii) Solve $x < \sqrt{x+2}$: $x < 2$

(iii) Claim: $\forall n$ $x_n < 2$.

Basis: $x_1 = 1 < 2$.

If $n > 1$, $x_n = \sqrt{x_{n-1} + 2} < \sqrt{2+2} = 2$ ☺

(iv) (x_n) is strictly increasing

Since $x_n < 2$, $x_{n+1} = \sqrt{x_n + 2} > x_n$ ☺

(v) Since (x_n) is bdd above & increasing, $\exists x$ $x_n \to x$.

Take limit of $x_{n+1} = \sqrt{x_n + 2}$: $x = \sqrt{x+2}$

$\therefore x = 2$ so $x_n \to 2$ ☺

Alt. Proof of (iv) Induction: Basis $x_2 = \sqrt{1+2} = \sqrt{3} > x_1 = 1$

If $n > 2$ $x_n - x_{n-1} = \sqrt{x_{n-1} + 2} - \sqrt{x_{n-2} + 2}$

$\therefore x_{n-1} - x_{n-2} > 0 \Rightarrow x_n - x_{n+1}$ ($\sqrt{x+2}$ is an increasing function)