1. Let $c \in \mathbf{Q}$ and $C = \{r \in \mathbf{Q}\colon r > c\}$.

   (a) Prove that $C$ is a Dedekind cut ($C$ represents the real number $c$).

   (b) Suppose $D$ is a Dedekind cut. Prove that $D < C$ if and only if $c \in D$.

   Hint: $D < C \Leftrightarrow C$ is a proper subset of $D$.

Dedekind cuts are nonempty proper rays (to the right) without a min.

a) $c+1 \in \mathbb{Q}$, $c+1 > c$, so $c+1 \in C$, so $C \neq \emptyset$

$c \not> c$, so $c \notin C$, so $C \neq \mathbb{Q}$ (proper)

Ray: Let $r \in C$ and $s \geq r$, Since $r > c$, $s > c$, so $s \in C$ ☺

Let $r \in C$, then $r > c$, so $r > \underbrace{\frac{r+c}{2}}_{\in \mathbb{Q}} > c$

so $\frac{r+c}{2} \in C$, so $r$ is not min $C$. (density)

$\therefore$ $C$ has no min $\therefore$ $C$ is a Dedekind cut

b) $D < C \Leftrightarrow C$ is a proper subset of $D$

$(C \subseteq D \wedge C \neq D)$

"$\Rightarrow$" Suppose $D < C$, then $C \neq D$, so

$\exists r \in D \setminus C$

Since $r \notin C$, $r \not> c$, $r \leq c$

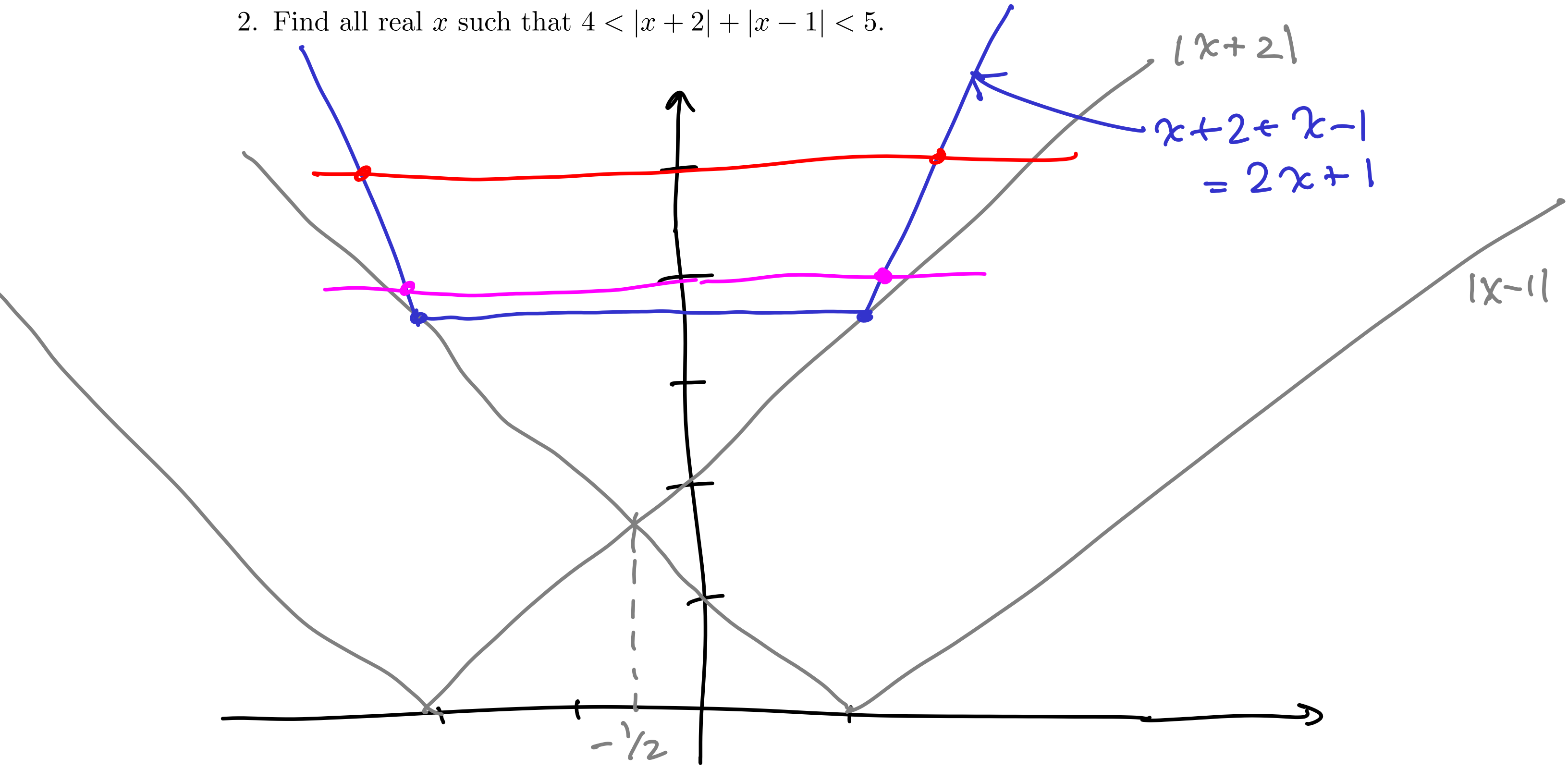
Since $r \in D$ and $D$ is a ray and $c \geq r$, $c \in D$ ☺

"$\Leftarrow$" Suppose $c \in D$, since $c \not> c$, $c \notin C$.

Given $r \in C$, $r > c$, so since $c \in D$, (ray) $r \in D$ $\therefore C \subseteq D$

Since $c \in D \setminus C$, $C$ is a proper subset of $D$ ☺

2. Find all real $x$ such that $4 < |x+2| + |x-1| < 5$.




$2x+1=5 \Rightarrow x=2$

$2x+1=4 \Rightarrow x=\frac{3}{2}$

reflect w.r.t. $-\frac{1}{2}$ $\quad x=-3$

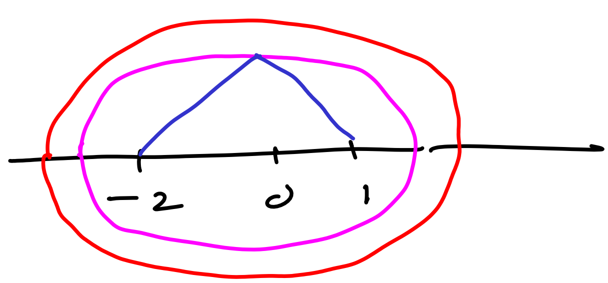
(or solve $2x+1=-5$, $2x+1=-4$) $\quad x=-\frac{5}{2}$

$$\boxed{\left(-3,-\frac{5}{2}\right) \cup \left(\frac{3}{2}, 2\right)}$$

In $\mathbb{C}$:

Elliptical annulus

Foci: 1, -2



3. For each of sup/inf/max/min either find it or state it doesn't exist for the set $\underbrace{\{1/n^2 \colon n \in \mathbf{N}\}}_{S}$. Prove your assertions.

$S = \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \dots\}$

$\max S = 1$ (so $\sup S = 1$) $\quad \inf S = 0$ (no min)

$1 \in S$ and $\forall n \in \mathbb{N}$ $\frac{1}{n^2} \leq 1$, so $1 = \max S$.

$\forall n \in \mathbb{N}$ $\frac{1}{n^2} > 0$, so $0$ is a lower bound for $S$.

If $u > 0$, by Archimedean property

$\exists n \in \mathbb{N}$ $n > \frac{1}{\sqrt{u}}$, then $\frac{1}{n} < \sqrt{u}$

so $\frac{1}{n^2} < u$, so $u$ is not a lower bd for $S$

$\therefore \inf S = 0$.

Since $0 \notin S$. $0$ is not $\min S$, so no min.

4. Suppose $A, B$ are nonempty bounded subsets of $\mathbf{R}$. Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

WLOG assume $\sup A \geqslant \sup B$

In this case $\sup A = \max\{\sup A, \sup B\}$

Given $x \in A \cup B$, $x \in A$ or $x \in B$

If $x \in A$, $x \leq \sup A$

If $x \in B$, $x \leq \sup B \leq \sup A$

$\therefore$ max is an upper bound for $A \cup B$

Let $u < \sup A \ (= \max)$. Then $u$ is not an upper Bound for $A$, so $\exists\, a \in A$ $a > u$.

Since $a \in A \cup B$, $u$ is not an upper bound for $A \cup B$, so $\sup(A \cup B) = \max$ :)

<u>Alt:</u> $A \subseteq A \cup B$, so $\sup A \leq \sup(A \cup B)$

Similarly $\sup B \leq \sup(A \cup B)$

$\therefore$ max $\leq \sup(A \cup B)$

(still need the other direction)

$$\frac{n}{n+1} = \frac{1}{1+(1/n)} \to 1, \quad 1/n \to 0$$

5. Does the sequence $\dfrac{n}{n+1}$ converge? Prove your assertion. Same for the sequence $(-1)^n \dfrac{n}{n+1}$.

$$\frac{n}{n+1} = 1 - \frac{1}{n+1} \to 1, \quad \frac{1}{n+1} \to 0$$

✓ $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

&  $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \dots$ Diverges

Given $\varepsilon > 0$, By Archimedean property $\exists k \in \mathbb{N}$ $k > \frac{1}{\varepsilon}$. Then $\forall n \geq k$

$n+1 > n \geq k > \frac{1}{\varepsilon}$, so $\frac{1}{n+1} < \varepsilon$, so

$$\left| 1 - \frac{n}{n+1} \right| = \frac{1}{n+1} < \varepsilon \quad ☺$$

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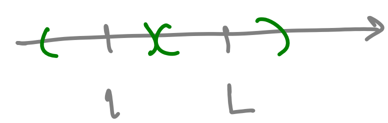
Suppose $(-1)^n \frac{n}{n+1} \to L$, then $L \neq 1$ or $L \neq -1$.

Case $L \neq -1$ is similar to $L \neq 1$, since $-\frac{n}{n+1} \to -1$, so assume $L \neq 1$. Pick $\varepsilon > 0$ such that

$$V_\varepsilon(L) \cap V_\varepsilon(1) = \emptyset$$




For example, let $\varepsilon = \dfrac{|L-1|}{2}$

Since $(-1)^n \frac{n}{n+1} \to L$, $\exists k_1$ $\forall n \geq k_1$ $(-1)^n \frac{n}{n+1} \in V_\varepsilon(L)$

Since $\frac{n}{n+1} \to 1$, $\exists k_2$ $\forall n \geq k_2$ $\frac{n}{n+1} \in V_\varepsilon(1)$

For any $n \geq \max\{k_1, k_2\}$, $n$ - even

$$\frac{n}{n+1} \in V_\varepsilon(L) \cap V_\varepsilon(1) \quad ☹$$

**Alt:** Lemma If $x_n \to L$, then $|x_{n+1} - x_n| \to 0$

Pf Given $\varepsilon > 0$, $\exists k$ $\forall n \geq k$ $|x_n - L| < \frac{\varepsilon}{2}$

Since $n+1 > n$, also $|x_{n+1} - L| < \frac{\varepsilon}{2}$

$$|x_{n+1} - x_n| \leq |x_{n+1} - L| + |L - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

:)

**Alt proof:** $x_{n+1} - x_n \xrightarrow{\text{properties of limits.}} L - L = 0$

so $|x_{n+1} - x_n| \to 0$ :)

$$\left| (-1)^n \frac{n}{n+1} - (-1)^{n+1} \frac{n+1}{n+2} \right| = \frac{n}{n+1} + \frac{n+1}{n+2} \to 1 + 1 = 2$$

$$\not\to 0$$ :(