1. Suppose C and D are Dedekind cuts. Prove that their intersection $C \cap D$ is a Dedekind cut. Give a concrete example of a sequence of Dedekind cuts (D_n) whose intersection is not a Dedekind cut.

(i) Since $C \cap D \subseteq C \neq \mathbb{Q}$, $C \cap D \neq \mathbb{Q}$

(ii) Since $C, D \neq \phi$ $\exists\, c \in C$, $d \in D$

WLOG assume $c \leq d$. Then $d \in C$, so $d \in C \cap D$.

$\therefore C \cap D \neq \phi$

(iii) If $x \in C \cap D$ and $y \geqslant x$, since $x \in C$, $y \in C$

and since $x \in D$, $y \in D$, so $y \in C \cap D$

$\therefore C \cap D$ is a ray (to the right)

(iv) Suppose $x = \min(C \cap D)$.

Rays are linearly ordered (C, D rays $\Rightarrow C \subseteq D$ or $D \subseteq C$)

Pf Suppose C & D are rays and $C \not\subseteq D$, $D \not\subseteq C$

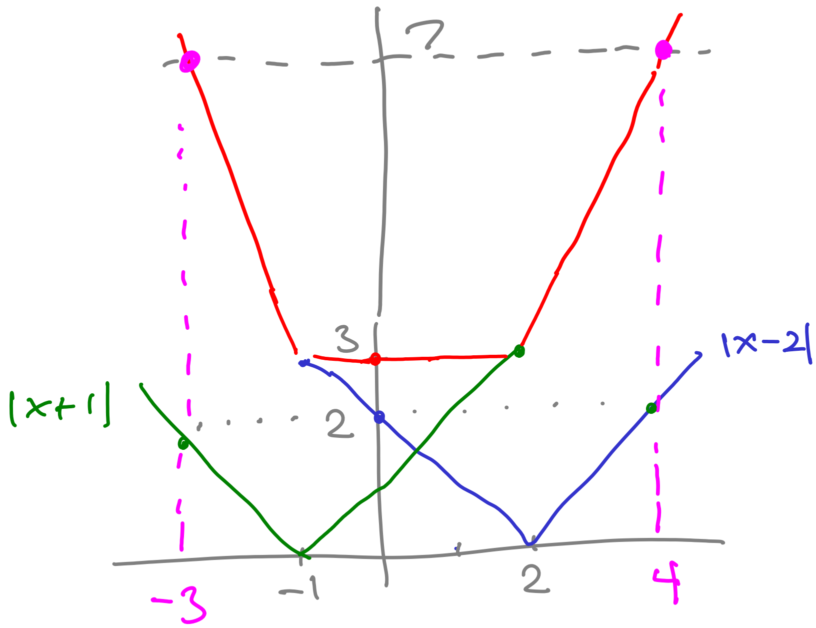
Then $\exists\, c \in C \setminus D$ and $d \in D \setminus C$

WLOG assume $c \leq d$. Then $d \in C$ ☹

WLOG assume $D \subseteq C$. Then $C \cap D = D$,

So $x = \min D$ ☹

2. Find all real x such that $3 < \underbrace{|x-2| + |x+1|}_{f} < 7$.



$$f = \begin{cases} -x-1-x+2 & \text{for } x<-1 \\ x+1-x+2 & \text{for } -1 \le x < 2 \\ x+1+x-2 & \text{for } 2 \le x \end{cases}$$

$$= \begin{cases} -2x+1 & \text{for } x<-1 \\ 3 & \text{for } -1 \le x < 2 \\ 2x-1 & \text{for } 2 \le x \end{cases}$$

$$-2x+1 = 7$$
$$-2x = 6$$
$$x = -3$$

$$2x-1 = 7$$
$$2x = 8$$
$$x = 4$$

Final answer: $-3 < x < -1$ or $2 < x < 4$

i.e. $x \in (-3,-1) \cup (2,4)$

3. Suppose A, B are nonempty bounded subsets of $\mathbf{R}$ that are not disjoint. Prove that $\inf(A \cap B) \geq \min\{\inf A, \inf B\}$. Give a concrete example where the inequality is strict.

Since $A \cap B \subseteq A$ and A is bdd, so is $A \cap B$

and $\inf(A \cap B) \geq \inf A$

Similarly $\inf(A \cap B) \geq \inf B$

$\therefore \inf(A \cap B) \geq \min\{\inf A, \inf B\}$

If $A = (-1, 0]$, $B = [0, 1)$, then

$A \cap B = \{0\}$.

$\inf(A \cap B) = 0$

$\inf A = -1$

$\inf B = 0$

$$\therefore \underbrace{\inf(A \cap B)}_{0} > \underbrace{\min\{\inf A, \inf B\}}_{-1}$$

4. Suppose (x_n) is sequence in $\mathbf{R}$ that is not bounded. Prove that (x_n) has a subsequence convergent to $+\infty$ or a subsequence convergent to $-\infty$.

If (x_n) is not bounded, then (x_n) is not bdd above or (x_n) is not bdd below.

Suppose (x_n) is not bdd above.

Claim Every tail of (x_n) is not bdd above.

Pf Suppose $\{x_n, n \geq k\}$ is bdd above by M

Then (x_n) is bdd above by $\max\{M, x_1, \dots x_{k-1}\}$ $\ddot{\frown}$

1 is not an upper bd for (x_n), so

$\exists n_1 \quad x_{n_1} > 1$

2 is not an upper bd for $\{x_n : n > n_1\}$

So $\exists n_2 > n_1 \quad x_{n_2} > 2$

3 is not an upper bd for $\{x_n : n > n_2\}$

So $\exists n_3 > n_2 \quad x_{n_3} > 3$ etc.

Since $n_1 < n_2 < n_3 \dots$ we have a subsequence

Note: $\forall k \;\; x_{n_k} > k > 0 \qquad 0 < \dfrac{1}{x_{n_k}} < \dfrac{1}{k}$

By squeeze law $\dfrac{1}{x_{n_k}} \to 0$ (0^+)

So $x_{n_k} \to +\infty$ $\ddot{\smile}$

Similarly if (x_n) is not bdd below, $\exists$ subseq $\to -\infty$ $\ddot{\smile}$

5. Find lim sup and lim inf of the sequence $x_n = (-1)^n - \frac{1}{n}$. Prove your assertion for lim inf.

k-th tail $\quad S_k = \{x_n : n \geq k\} = (-1)^k - \frac{1}{k}, \ (-1)^{k+1} - \frac{1}{k+1}, \ldots$

If k is even $\quad S_k = 1 - \frac{1}{k}, \ -1 - \frac{1}{k+1}, \ 1 - \frac{1}{k+2}, \ldots$

Then $\quad \sup S_k = 1$

Pf $\forall k \quad S_k < 1$, so 1 is an upper bd.

If $r < 1$, by Archimedean property

$\exists n > \max\{k, \frac{1}{1-r}\}$. then $\frac{1}{n} < 1-r$,

so $-\frac{1}{n} > r-1$, so $1 - \frac{1}{n} > 1 + r - 1 = r$

$\therefore r$ is not an upper bd ☺

$\therefore \quad \limsup x_n = \lim 1 = 1$

$\inf S_k = -1 - \frac{1}{k+1}$

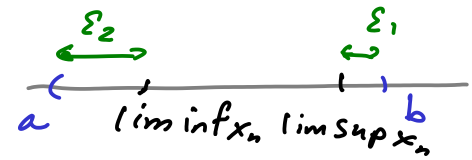
$\therefore \liminf x_n = \lim\left(-1 - \frac{1}{k+1}\right) = -1$ ☺

If k is odd the proof is similar

6. Suppose (x_n) is a bounded sequence and $\limsup x_n$ and $\liminf x_n$ belong to an open interval (a, b). Prove that $\exists k \in \mathbf{N}\ \forall n \in \mathbf{N}\ n \geq k \Rightarrow x_n \in (a, b)$.

Let $S_m = \{x_n : n \geq m\}$ be the m-th tail.

Let $\varepsilon_1 = b - \limsup x_n$



Since $\sup S_m \to \limsup x_n$, $\exists k_1$

$\forall m \geq k_1\ \limsup x_n - \varepsilon_1 < \sup S_m < \limsup x_n + \varepsilon_1$

$= \limsup x_n + b - \limsup x_n = b$

Similarly, let $\varepsilon_2 = \liminf x_n - a$

Since $\inf S_m \to \liminf x_n$, $\exists k_2$

$\forall m \geq k_2\ \ \underbrace{\liminf x_n - \varepsilon_2}_{= \liminf x_n - \liminf x_n + a = a} < \inf S_m < \liminf x_n + \varepsilon_2$

For $m \geq \max\{k_1, k_2\}$, if $n \geq m$, then $x_n \in S_m$,

so $a < \inf S_m \leq x_n \leq \sup S_m < b$

i.e. $x_n \in (a, b)$

7. Prove that every convergent sequence in $\mathbf{R}$ is Cauchy.

Suppose $x_n \to L$

Given $\varepsilon > 0$ $\exists k$ $\forall n \geq k$ $|x_n - L| < \frac{\varepsilon}{2}$

If $m, n \geq k$ $|x_n - x_m| = |x_n - L + L - x_m|$

$\leq |x_n - L| + |L - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

☺

8. Suppose $\sum x_n$ is convergent. Prove that the sequence (x_n) converges to 0.

Let $s_k = \sum_{n=1}^{k} x_n$, then s_k converges.

Say $s_k \to S$. Also $s_{k-1} \to S$

So $a_k = s_k - s_{k-1} \to S - S = 0$

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