① Suppose $\delta > 0$ and $|x-1| < \delta$

$$-\delta < x - 1 < \delta$$

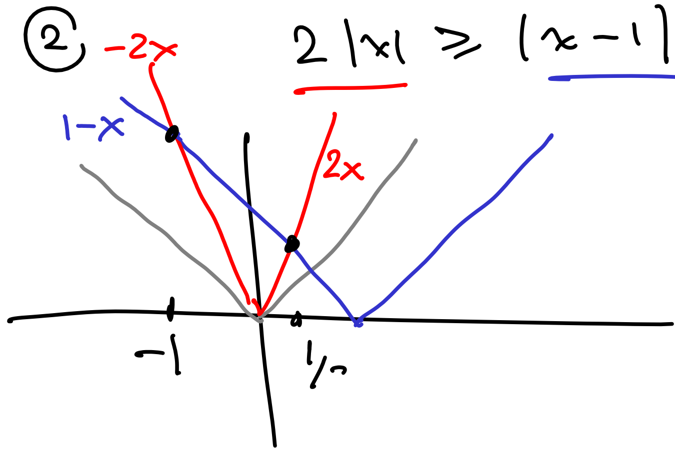
$$-2-\delta < 2-\delta < x+1 < 2+\delta$$

$$|x+1| < 2+\delta$$

$$|x^2-1| = |(x-1)(x+1)| = |x-1||x+1| < \delta(2+\delta)$$

☺

② $\underline{2|x|} \geq \underline{|x-1|}$



$$-2x = 1 - x$$

$$x = -1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$

$x \leq -1$ or $x \geq \frac{1}{3}$

$$\underline{(-\infty, -1) \cup (\tfrac{1}{3}, \infty)}$$

③ Since $A \neq \emptyset$, $\exists a \in A$

Since $\inf A \leq a \leq \sup A$ and $\inf A = \sup A$

$\inf A = a = \sup A$.

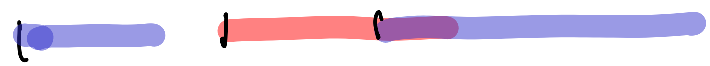
Since this is true for any element of A,

$A = \{a\}$ ☺

④ Suppose A, B - bdd, $A \cap B \neq \emptyset$

Prove $A \cap B$ is bdd below and

$\inf(A \cap B) \geq \max\{\inf A, \inf B\}$



Note: $A \cap B \neq \emptyset \Rightarrow A \neq \emptyset, B \neq \emptyset$

$\inf A$ is a lower bound for A.

Since $A \cap B \subseteq A$, $\inf A$ is a lower bound for $A \cap B$.

Pf Given $x \in A \cap B$, $x \in A$, so

in $\inf A \leq x$. ☺

$\therefore \inf A \cap B \geq \inf A$

Similarly $\inf A \cap B \geq \inf B$

$\therefore \inf(A \cap B) \geq \max\{\inf A, \inf B\}$

Note: Let $A = (0,1) \cup (3,5)$ $\inf A = 0$

$B = (2,4)$ $\inf B = 2$

then $A \cap B = (3,4)$ $\inf(A \cap B) = 3$

(see pic above)

(5) $A, B \subseteq \mathbb{R}^+$, nonempty, bdd.

$C = AB = \{x : \exists a \in A, b \in B \quad x = ab\}$

(all possible products of element of A and B)

Want: $\sup C = \sup A \cdot \sup B$

1. Given $x \in C$, $\exists a \in A, b \in B \quad x = ab$

Since $a \le \sup A$, $b \le \sup B$

Since $b > 0$, $ab \le \sup A \cdot b$

Since $A \ne \phi$, $\exists a \in A$, Since $A \subseteq \mathbb{R}^+$, $a > 0$

$\sup A \ge a > 0$, so $\sup A > 0$

So since $b \le \sup B$, $\sup A \cdot b \le \sup A \sup B$

Now $x = ab \le \sup A \cdot b \le \sup A \sup B$

$\therefore$ $\sup A \sup B$ is an upper bound for C.

Suppose $u > 0$, $\quad u < \underbrace{\sup A}_{>0} \cdot \underbrace{\sup B}_{>0}$

<u>Lemma</u> if $0 < u < pq$, then $\exists \varepsilon > 0$

s.t. $u < \underbrace{(p-\varepsilon)(q-\varepsilon)}_{pq-(p+q)\varepsilon+\varepsilon^2} < pq$

<u>Pf</u>

$u < pq - (p+q)\varepsilon + \varepsilon^2$

$(p+q)\varepsilon - \varepsilon^2 < pq - u$

If $\varepsilon = \dfrac{pq-u}{p+q} (>0)$

then $(p+q)\varepsilon - \varepsilon^2 < (p+q)\varepsilon = (p+q)\dfrac{pq-u}{p+q} =$

$= pq-u$ ☺

Continuing with proof. By Lemma

$\exists\, \varepsilon>0 \qquad u < \underbrace{(\sup A - \varepsilon)}_{\text{not an upper bd for } A}\underbrace{(\sup B - \varepsilon)}_{\text{not an upper bound for } B}$

$\exists\, a \in A,\ b \in B \qquad a > \sup A - \varepsilon,\ b > \sup B - \varepsilon$

then $u < ab$, $\therefore u$ is not an upper bd for C

☺

Extra credit w/o assuming "positive"

$\sup C = \max\{\sup A \sup B,\ \inf A \inf B\}$

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Since $D_1, D_2 \neq \emptyset$

$\exists\, x_1 \in D_1,\ x_2 \in D_2$

WLOG assume $x_1 \geq x_2$

Since D_2 is a ray $x_1 \in D_2$

$\therefore x_1 \in D_1 \cap D_2 \quad \therefore D_1 \cap D_2 \neq \emptyset$:)

b) False, however; $D \geq 0 \Leftrightarrow 0 \notin D$

Dedekind cut of 0: $\{r \in \mathbb{Q} : r > 0\}$

$D \geq 0 \Leftrightarrow D \subseteq \{r \in \mathbb{Q} : r > 0\}$

if $D \geq 0$, then $\forall r \in D,\ r > 0$, $\therefore 0 \notin D$

if $D < 0$, $\{r \in \mathbb{Q}, r > 0\} \subsetneq D$

Then $\exists\, x \in D,\ x \notin \{r \in \mathbb{Q} : r > 0\}$

i.e. $x \leq 0$

Since D is a ray, $0 \in D$:)