Midterm 2 / 2023.11.2 / MAT 3013.002 / Foundations of Mathematics

1. Prove by induction that $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$ for $n = 1, 2, \ldots$

Basis: $n=1$: $\quad$ L.H.S. $= \dfrac{1}{(2-1)(2+1)} = \dfrac{1}{3}$ $\qquad$ R.H.S $= \dfrac{1}{2+1} = \dfrac{1}{3}$

For $n>1$ assume $\displaystyle\sum_{k=1}^{n-1} \frac{1}{(2k-1)(2k+1)} = \frac{n-1}{2(n-1)+1} = \frac{n-1}{2n-1}$

Then $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n-1} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2n-1)(2n+1)}$

$$= \frac{n-1}{2n-1} + \frac{1}{(2n-1)(2n+1)} = \frac{(n-1)(2n+1)+1}{(2n-1)(2n+1)} = \frac{2n^2+n-2n\cancel{-1}\cancel{+1}}{(2n-1)(2n+1)}$$

$$= \frac{2n^2-n}{(2n-1)(2n+1)} = \frac{n\cancel{(2n-1)}}{\cancel{(2n-1)}(2n+1)} = \frac{n}{2n+1}$$

2. Prove by induction that $5^n \geq 1 + 4^n$ for $n = 1, 2, \ldots$

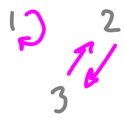
Basis: $n=1$ $\quad$ L.H.S. $= 5$ $\quad$ R.H.S. $= 1+4 = 5$

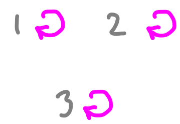
For $n>1$ assume $5^{n-1} \geq 1 + 4^{n-1}$

Then $5^n = 5^{n-1}\cdot 5 \geq (1+4^{n-1})5 = 5 + 4^{n-1}\cdot 5 > 1 + 4^{n-1}\cdot 4 = 1 + 4^n$

3. Let $A = \{1, 2, 3\}$ and let $R = \{[1, 1], [2, 3], [3, 2]\}$ be a relation on A. Find $R \circ R$ and $R \circ R \circ R$ and sketch a digraph for each of the relations $R, R \circ R, R \circ R \circ R$

$R \circ R = \{ [1,1], [2,2], [3,3] \} = I_A$, so $R \circ R \circ R = R$

R : 

$R \circ R$: 

4. Define a relation R on $\mathbf{R} \times \mathbf{R}$ by $[x, y]R[r, s] \Leftrightarrow x - y = r - s$. Prove that R is an equivalence relation. On the same set of axes sketch the equivalence class of $[2, 2]$ and the equivalence class of $[2, 3]$

Reflexive : if $[x, y] = [r, s]$, then $x = r$ and $y = s$, so $x - y = r - s$ ☺

Symmetric : by inspection ☺

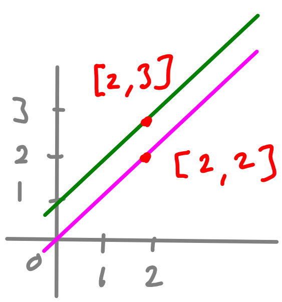
Transitive : if $x - y = r - s$ and $r - s = u - v$, then $x - y = u - v$ ☺

$[x, y] \in [[2,2]] \Leftrightarrow [x,y] R [2,2] \Leftrightarrow x - y = 2 - 2 \Leftrightarrow y = x$

Thus, $[[2,2]] = \{ [x,y] \in \mathbb{R} \times \mathbb{R} : y = x \}$

$[x, y] \in [[2,3]] \Leftrightarrow [x,y] R [2,3] \Leftrightarrow x - y = 2 - 3 \Leftrightarrow y = x + 1$

Thus, $[[2,3]] = \{ [x,y] \in \mathbb{R} \times \mathbb{R} : y = x + 1 \}$



5. Explain why the set of all even integers $2\mathbf{Z}$ and the set of all odd integers $1+2\mathbf{Z}$ form a partition of $\mathbf{Z}$. Describe the equivalence relation on $\mathbf{Z}$ whose quotient set is the above partition $\{2\mathbf{Z}, 1+2\mathbf{Z}\}$

Cover: $\mathbb{Z} \subseteq 2\mathbb{Z} \cup (1+2\mathbb{Z})$

If $n \in \mathbb{Z}$, then n is even or n is odd

Proof: By the Division Algorithm

if $n \in \mathbb{Z}$, then $\exists!\, q, r \quad n = 2q + r \wedge 0 \leq r < 2 \quad (r = 0 \text{ or } 1)$

If $r = 0$, then $n = 2q \in 2\mathbb{Z}$. If $r = 1$, then $n = 2q+1 \in 1+2\mathbb{Z}$

so $n \in 2\mathbb{Z} \cup (1+2\mathbb{Z})$ ☺

Pairwise disjoint: $2\mathbb{Z} \cap (1+2\mathbb{Z}) = \emptyset$

If $n \in \mathbb{Z}$, then n cannot be both even and odd

Proof: If $2\mathbb{Z} \cap (1+2\mathbb{Z}) \neq \emptyset$, then $\exists\, n \in 2\mathbb{Z} \cap (1+2\mathbb{Z})$

Then $n \in 2\mathbb{Z}$, so $\exists\, q_1 \in \mathbb{Z} \quad n = 2q_1$ and $n \in 1+2\mathbb{Z}$,

so $\exists\, q_2 \in \mathbb{Z} \quad n = 1 + 2q_2$, so $2q_1 = 1 + 2q_2$, so $1 = 2(q_1 - q_2)$ ☹

Suppose R is an equivalence relation on $\mathbb{Z}$

such that $\mathbb{Z}/R = \{2\mathbb{Z}, 1+2\mathbb{Z}\}$.

(in other words the equivalence classes of R are $2\mathbb{Z}$ and $1+2\mathbb{Z}$)

If $m, n \in \mathbb{Z}$, then

$m R n \iff (m \in 2\mathbb{Z} \wedge n \in 2\mathbb{Z}) \vee (m \in 1+2\mathbb{Z} \wedge n \in 1+2\mathbb{Z})$

$\iff$ m and n are even or m and n are odd

$\iff$ m and n have the same parity ☺