Midterm 2 / 2023.3.31 / MAT 3013.001 / Foundations of Mathematics

1. Let x_n be the sequence of integers recursively defined by

$$x_0 = 0$$
$$x_1 = -5$$
$$x_n = 7x_{n-1} - 6x_{n-2} \text{ for } n > 1$$

Prove by induction on n that $x_n = 1 - 6^n$ for all $n \geq 0$

Basis: $n=0$: $1-6^n = 1-1 = 0$, $n=1$: $1-6^1 = 1-6 = -5$ ☺

For $n>1$ assume $0 \leq k < n \Rightarrow x_k = 1-6^k$

In particular $x_{n-1} = 1-6^{n-1}$, $x_{n-2} = 1-6^{n-2}$

$$x_n = 7x_{n-1} - 6x_{n-2} = 7(1-6^{n-1}) - 6(1-6^{n-2})$$

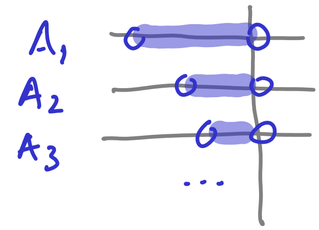
$$= 7 - 7\cdot 6^{n-1} - 6 + \underbrace{6\cdot 6^{n-2}}_{6^{n-1}} = 1 - 6\cdot 6^{n-1} = 1-6^n$$ ☺

2. For each natural number $n > 0$ let A_n be the interval $(-\frac{1}{n}, 0)$

(a) Find the union $\bigcup_{n=1}^{\infty} A_n$ and the intersection $\bigcap_{n=1}^{\infty} A_n$ of this family of sets.

(b) Prove your assertions.

$\bigcup_{n=1}^{\infty} A_n = A_1 = (-1, 0)$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$



Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then $\exists n \in \mathbb{N}$ $x \in (-\frac{1}{n}, 0)$

Since $n \geq 1$, $\frac{1}{n} \leq 1$, $-\frac{1}{n} \geq -1$, so $x \in A_1$

Conversely if $x \in A_1$, $x \in \bigcup_{n=1}^{\infty} A_n$ ☺

Suppose $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Then $\exists x \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $x \in A_n$,

i.e. $-\frac{1}{n} < x < 0$. By the Archimedean principle

$\exists n \in \mathbb{N}$ $n > -\frac{1}{x}$. Then $\frac{1}{n} < -x$, so $-\frac{1}{n} > x$

so $x \notin A_n$, so $x \notin \bigcap_{n=1}^{\infty} A_n$ ☹

3. Define a relation S on the real line $\mathbf{R}$ by $aSb \Leftrightarrow a-b$ is an integer multiple of 2π

(a) Prove that S is an equivalence relation.

(b) Describe the equivalence classes.

(c) Extra credit: Explain why the quotient set $\mathbf{R}/S$ (the set of all equivalence classes) is in one-to-one correspondence with the unit circle.

(a) Reflexive: If $a \in \mathbb{R}$, $a-a=0=0\cdot 2\pi$, so aSa

Symmetric: If $a,b\in\mathbb{R}$ aSb, $\exists n \in \mathbb{Z}$ $a-b=2n\pi$.

then $b-a=-(a-b)=-2n\pi=2(-n)\pi$, so bSa

Transitive: If $aSb \wedge bSc$, $\exists m,n\in\mathbb{Z}$ $a-b=2m\pi \wedge b-c=2n\pi$

Then $a-c=a-b+b-c=2m\pi+2n\pi=2\underbrace{(m+n)}_{\in\mathbb{Z}}\pi$, so aSc

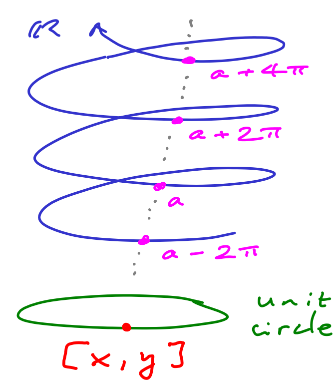
(b) $[a]=\{b\in\mathbb{R} : bSa\}=\{b\in\mathbb{R} : \exists n\in\mathbb{Z}\ b-a=2n\pi\}=$

$=\{b\in\mathbb{R} : \exists n\in\mathbb{Z}\ b=a+2n\pi\}=$

$=\{a+2n\pi : n\in\mathbb{Z}\}=\{\dots, a-2\pi, a, a+2\pi, a+4\pi, \dots\}$

(c) Two angles give the same point on the unit circle $\Longleftrightarrow$ they differ by an integer multiple of 2π

$$\left.\begin{array}{l} x(t)=\cos t \\ y(t)=\sin t \end{array}\right\} \rightarrow$$



4. For each of the following relations S on $\mathbf{R}$, determine whether S is reflexive, whether S is symmetric, and whether S is transitive. Explain.

(a) $aSb \Leftrightarrow ab = 1$

(b) $aSb \Leftrightarrow ab \geq 0$

(a) Not reflexive $2 \cdot 2 \neq 1$

Symmetric by inspection, since real multiplication is commutative

Not transitive: $2 \cdot \frac{1}{2} = 1$, $\frac{1}{2} \cdot 2 = 1$, but $2 \cdot 2 \neq 1$

(b) Reflexive: $a \cdot a = a^2 \geq 0$

Symmetric by inspection, since real multiplication is commutative

Not transitive: $1 \cdot 0 = 0 \geq 0$, $0 \cdot (-1) = 0 \geq 0$, but $1 \cdot (-1) = -1 < 0$