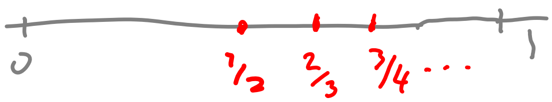
Final exam / 2023.5.10 / MAT 3013.001 / Foundations of Mathematics

1. Let $S = \left\{ \frac{n}{n+1} : n \in \mathbf{N} \right\} \subseteq \mathbf{R}$

Does S have a sup? inf? max? min? If so, find them. Prove your assertions.



Note: $\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$

$\min S = \frac{1}{2}$ (so $\inf S = \frac{1}{2}$)

Pf: (i) $\frac{1}{2} \in S$ (ii) if $1 - \frac{1}{n+1} \in S$, then $1 - \frac{1}{n+1} \geq \frac{1}{2}$

Work: $n \geq 1$, $n+1 \geq 2$, $\frac{1}{n+1} \leq \frac{1}{2}$, $-\frac{1}{n+1} \geq -\frac{1}{2}$, $1 - \frac{1}{n+1} \geq 1 - \frac{1}{2} = \frac{1}{2}$

S has no max

Pf: Suppose $1 - \frac{1}{n+1} \in S$, then $1 - \frac{1}{(n+1)+1} = 1 - \frac{1}{n+2} \in S$ and $1 - \frac{1}{n+1} < 1 - \frac{1}{n+2}$

Work: $1 < 2$, $n+1 < n+2$, $\frac{1}{n+1} > \frac{1}{n+2}$, $-\frac{1}{n+1} < -\frac{1}{n+2}$, $1 - \frac{1}{n+1} < 1 - \frac{1}{n+2}$

$\sup S = 1$

Pf: (i) upper bound: $1 - \frac{1}{n+1} < 1$

Work: $n+1 > 0$ so $\frac{1}{n+1} > 0$ so $-\frac{1}{n+1} < 0$ so $1 - \frac{1}{n+1} < 1$

(ii) least: Let $b < 1$ Want: b is not an upper bound

Want: n s.t. $1 - \frac{1}{n+1} > b$

Scratch work: $-\frac{1}{n+1} > b - 1$, $\frac{1}{n+1} < \underbrace{1-b}_{>0}$, $n+1 > \frac{1}{1-b}$, $n > \frac{1}{1-b} - 1$

By the Archimedean principle, $\exists n \in \mathbb{N}$ $n > \frac{1}{1-b} - 1$

so by scratch work $1 - \frac{1}{n+1} > b$:)

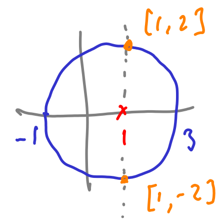
2. Determine whether each of the following relations $S: \mathbf{R}\to\mathbf{R}$ is a function. Prove your assertions.

(a) $S = \{[x,y]\in\mathbf{R}^2\colon (x-1)^2+y^2=4\}$

(b) $S = \{[x,y]\in\mathbf{R}^2\colon \underbrace{|y|<1}_{y<1 \,\wedge\, y>-1}\}$

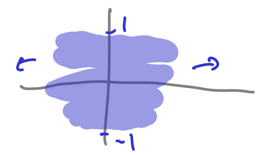
a) NOT A Function

Vertical line test: $[1,2]\,,\ [1,-2] \in S$



b) NOT A Function

Vertical line test: $[0,\tfrac{1}{2}]\,,\ [0,0] \in S$



3. Define $f: \mathbf{R}^2\to\mathbf{R}$ by $f(x,y)=x+2y$

(a) Prove that f is onto.

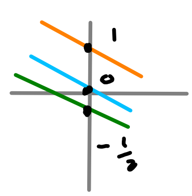
(b) Sketch the fibers $f^{-1}(\{-1\}), f^{-1}(\{0\}), f^{-1}(\{2\})$ on the same graph.

a) f is onto: Let $z\in\mathbb{R}$. Then $f(z,0)=z$

b) $f^{-1}(\{-1\}) = \{[x,y]\in\mathbb{R}^2 : x+2y=-1\}$ $\quad y = \dfrac{-x-1}{2}$

$f^{-1}(\{0\}) = \{[x,y]\in\mathbb{R}^2 : x+2y=0\}$

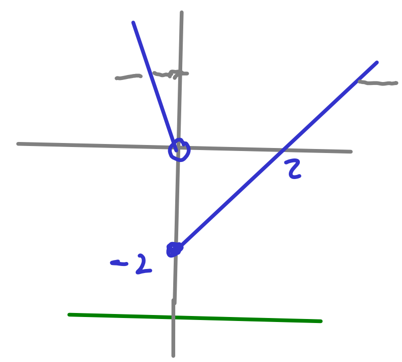
$f^{-1}(\{2\}) = \{[x,y]\in\mathbb{R}^2 : x+2y=2\}$



4. Define $f: \mathbf{R} \to \mathbf{R}$ by $f(x) = \begin{cases} -2x & \text{for } x < 0 \\ x - 2 & \text{for } x \geq 0 \end{cases}$

(a) Prove that f is not onto.

(b) Find the following images and preimages: $f([-1,1])$, $f^{-1}([2,\infty))$



a) f is not onto

Pf: Horizontal line test at $y = -3$

b) $f([-1,1]) = (0,2] \cup [-2,-1]$

Pf: $[-1,1] = [-1,0) \cup [0,1]$

(forward images preserve unions)

$f^{-1}([2,\infty)) = (-\infty,-1] \cup [4,\infty)$

5. With f as in preceding problem, give concrete examples of subsets $E, D \subseteq \mathbf{R}$ such that $D \neq f^{-1}(f(D))$ and $E \neq f(f^{-1}(E))$

Let $D = \{-1\}$, then $f^{-1}(f(D)) = f^{-1}(\{2\}) = \{-1,4\}$

Let $E = \{-3\}$, then $f(f^{-1}(E)) = f(\emptyset) = \emptyset$