1. Construct a truth table to establish the equivalence of implication with its contrapositive. In other words, use a truth table to prove $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	T	F	F	T

$p \Rightarrow q$ & $\sim q \Rightarrow \sim p$ have the same truth values, so are equivalent.

2. Translate "everybody loves somebody sometime" into the formal language of predicate calculus. Negate it and translate the negation back into human language.

Let $p(x,y,t)$ denote x loves y at time t.

$$\forall x \ \exists y \ \exists t \ p(x,y,t)$$

Negation: $\exists x \ \forall y \ \forall t \ \sim p(x,y,t)$

Someone doesn't love anybody, anytime,

3. Show that for arbitrary sets A, B, C, D we have $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$ and provide a concrete counterexample to subset the other way.

"$\subseteq$" Let $[x,y] \in (A \times B) \cup (C \times D)$

Then $[x,y] \in A \times B \ \vee \ [x,y] \in C \times D$

If $[x,y] \in A \times B$, then $x \in A \ \wedge \ y \in B$,

So $x \in A \cup C \ \wedge \ y \in B \cup D$

So $[x,y] \in (A \cup C) \times (B \cup D)$ $\ddot{\smile}$

"$\not\supseteq$" Let $A = \{1\}$ $\quad A \times B = \{[1,2]\}$

$B = \{2\}$ $\quad C \times D = \{[3,4]\}$

$C = \{3\}$ $\quad (A \times B) \cup (C \times D) = \underline{\{[1,2],[3,4]\}}$

$D = \{4\}$

$A \cup C = \{1,3\}$, $B \cup D = \{2,4\}$

$(A \cup C) \times (B \cup D) = \underline{\{[1,2],[1,4],[3,2],[3,4]\}}$

$\therefore (A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$ $\ddot{\frown}$

4. For each $n \in \mathbf{N}$ let $A_n = \{x \in \mathbf{R}: 0 \le x \le 1/n\} = [0, 1/n]$. Find the union and the intersection of this family of sets. Prove your assertions.

(i) $\bigcup_{n=1}^{\infty} A_n = [0,1]$

Pf "$\subseteq$" Let $a \in \bigcup_{n=1}^{\infty} A_n$. Then $\exists k$ $a \in A_k$.

So $0 \le a \le \frac{1}{k}$, but $\frac{1}{k} \le 1$ so $0 \le a \le 1 = [0,1]$ ☺

"$\supseteq$" If $a \in [0,1]$, then $a \in A_1$, so $a \in \bigcup_{n=1}^{\infty} A_n$ ☺

(ii) $\bigcap_{n=1}^{\infty} A_n = \{0\}$

"$\subseteq$" Let $a \in \bigcap_{n=1}^{\infty} A_n$. Then $\forall k \in \mathbb{N}$ $a \in A_k$.

In particular: $a \ge 0$, so $a = 0 \lor a > 0$

If $a = 0$, $a \in \{0\}$, done.

If $a > 0$, Pick $k > \frac{1}{a}$ (Archimedean property)

Then $a > \frac{1}{k}$, so $a \notin A_k$ ☹

"$\supseteq$" $\forall n$ $0 \le 0 \le \frac{1}{n}$, so $\forall n$ $0 \in A_n$, so $0 \in \bigcap_{n=1}^{\infty} A_n$ ☺

5. Use the principle of mathematical induction to prove Faulhaber's formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Basis for induction $(n=1)$: $\quad 1^2 = \frac{1(1+1)(2+1)}{6}$ ✓

Inductive step: let $n > 1$ and assume

$$\forall m < n \qquad \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

In particular, for $m = n-1$ assume

$$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} = \frac{(n-1)n(2n-1)}{6}$$

Then

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)n(2n-1)}{6} + n^2$$

$$= \frac{n}{6}\left[\underbrace{\underbrace{(n-1)(2n-1) + 6n}_{2n^2-n-2n+1+6n}}_{2n^2+3n+1}\right]$$

Meanwhile:

$$\frac{n(n+1)(2n+1)}{6} = \frac{n}{6}\left[\underbrace{2n^2+2n+n+1}_{2n^2+3n+1}\right]$$

=

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