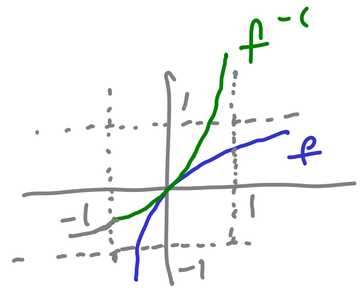
1. Let $f\colon \mathbf{R}\setminus\{-1\}\to\mathbf{R}$, $f(x)=\dfrac{x}{x+1}$.

   (a) Prove that f is not an increasing function on its domain, but its restrictions to intervals $f|_{(-\infty,-1)}$ and $f|_{(-1,\infty)}$ are strictly increasing.

   (b) Find a codomain for $f|_{(-1,\infty)}$ that makes the function bijective. Find the compositional inverse of our function. Sketch both our function and its inverse on the same set of axes.



a) $f(-2)=2>f(0)=0$.

$$f(x)=\frac{x}{x+1}=\frac{x+1-1}{x+1}=1-\frac{1}{x+1}$$

If $-1<x_1<x_2$, $0<x_1+1<x_2+1$, so

$$\frac{1}{x_1+1}>\frac{1}{x_2+1},\ \text{so}\ -\frac{1}{x_1+1}<-\frac{1}{x_2+1},\ \text{so}\ f(x_1)<f(x_2)$$

If $x_1<x_2<-1$, $x_1+1<x_2+1<0$, so

$$\frac{1}{x_1+1}>\frac{1}{x_2+1},\ \text{so}\ -\frac{1}{x_1+1}<-\frac{1}{x_2+1},\ \text{so}\ f(x_1)<f(x_2)$$

b) $x>-1 \iff y<1$, so pick $(-\infty,1)$ for codomain.

Let $y=1-\frac{1}{x+1}$. Then $\frac{1}{x+1}=1-y$, so $x+1=\frac{1}{1-y}$

so $x=\frac{1}{1-y}-1$. switch: $y=\frac{1}{1-x}-1=\frac{x}{1-x}$

Check: $1-\dfrac{1}{\frac{1}{1-x}-1+1}=1-(1-x)=x$

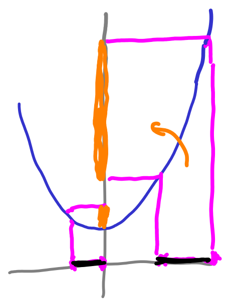
$$\frac{1}{1-\left(1-\frac{1}{x+1}\right)}-1=x+1-1=x$$

2. Let $f\colon \mathbf{R}\to\mathbf{R}$, $f(x) = x^2 + 1$. Find and sketch:

(a) $f([-1, 0] \cup [2, 4])$.

(b) $f^{-1}([-1, 5] \cup [17, 26])$.

a)

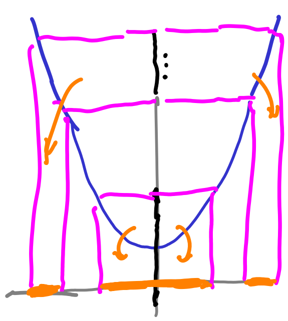


$f([-1,0]) = [1,2]$

$f([2,4]) = [5,17]$, so

$f([-1,0] \cup [2,4]) = [1,2] \cup [5,17]$

b)



$f^{-1}([-1,5]) = [-2,2]$

$f^{-1}([17,26]) = [-5,-4] \cup [4,5]$, so

$f^{-1}([-1,5] \cup [17,26]) = [-2,2] \cup [-5,-4] \cup [4,5]$

3. Suppose $f\colon A \to B$ is a function and R is a relation on A given by $xRy \Leftrightarrow f(x) = f(y)$.

   (a) Prove that R is an equivalence relation.

   (b) Prove that nonempty fibers of f are equivalence classes under R and *vice versa*.

a) (i) Reflexive: $\forall x \in A$ $f(x) = f(x)$, so xRx

(ii) Symmetric: If xRy, $f(x) = f(y)$, so yRx

(iii) Transitive: If $xRy \wedge yRz$, $f(x) = f(y) \wedge$
$f(y) = f(z)$, so $f(x) = f(z)$, so xRz

b) Suppose $f^{-1}(\{y\})$ is a nonempty fiber.
Then $\exists x \in A$, $f(x) = y$.

Further, $x' \in f^{-1}(\{y\}) \Leftrightarrow f(x') = y = f(x)$,
so $f^{-1}(\{y\}) = x/R$

Conversely, given $x \in A$, $x \in x/R$, so $x/R \neq \emptyset$
and is the fiber of $f(x)$.

4. Suppose $f\colon A \to B$ is a function and R is an equivalence relation on B with exactly two distinct equivalence classes $U, V \subseteq B$. Prove that $\{f^{-1}(U), f^{-1}(V)\}$ is a partition of A.

Since U, V are equivalence classes, they partition B, so

$U \cup V = B$ and $U \cap V = \emptyset$.

Let $x \in A$, then $f(x) \in B$, so $f(x) \in U \vee f(x) \in V$,

so $x \in f^{-1}(U) \wedge x \in f^{-1}(V)$

so $x \in f^{-1}(U) \cup f^{-1}(V)$

$\therefore A = f^{-1}(U) \cup f^{-1}(V)$.

If $x \in f^{-1}(U) \cap f^{-1}(V)$, $x \in f^{-1}(U) \wedge x \in f^{-1}(V)$

so $f(x) \in U \wedge f(x) \in V$, but $U \cap V = \emptyset$ $\ddot{\frown}$

$\therefore f^{-1}(U) \cap f^{-1}(V) = \emptyset$