Final MAT 3013.002 Sp. 2012

(1) For $n \geq 5$ $\quad 4^n > n^4$

Basis: $4^5 = 1024 > 5^4 = 625$

Inductive step: Suppose $n > 5$ and $4^k > k^4$ for all $5 \leq k < n$

$$4^n = 4^{n-1} \cdot 4 > (n-1)^4 \cdot 4 \geq n^4$$

Want

Scratch work:

$(n-1)^4 \cdot 4 \geq n^4$

$(n-1) 4^{1/4} \geq n$

$4^{1/4} n - 4^{1/4} \geq n$

$(4^{1/4} - 1) n \geq 4^{1/4}$

$n \geq \dfrac{4^{1/4}}{4^{1/4} - 1} \approx 3.4$ no problem ☺

Since $n \geq 5$, $n \geq \dfrac{4^{1/4}}{4^{1/4}-1}$ (≈ 3.4), so

$(4^{1/4} - 1) n \geq 4^{1/4}$, so $4^{1/4} n - 4^{1/n} \geq n$

so $(n-1) 4^{1/4} \geq n$, so $(n-1)^4 \cdot 4 \geq n^4$

so $4^n = 4^{n-1} \cdot 4 > (n-1)^4 \cdot 4 \geq n^4$ ☺

(2)

p	q	$p \to q$	$(p \to q) \to p$	$((p \to q) \to p) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	0
1	1	1	1	1

Neither a contradiction nor a tautology

p	q	$p \vee q$	$(p \vee q) \to q$	$((p \vee q) \to q) \to p$
0	0	0	1	0
0	1	1	1	0
1	0	1	0	1
1	1	1	1	1

Neither a contradiction nor a tautology

(3) $\exists x \, \forall y \, \exists z \, [\, p(x,y) \wedge \neg q(y,z) \;\vee\; \neg p(x,y) \wedge q(y,z) \,]$

(4) Zeros: $\{x \in \mathbb{R} : f(x) = 0\}$

crit pts: $\{x \in \mathbb{R} : f'(x) = 0\}$

Inflection pts: Crude: $\{x \in \mathbb{R} : f''(x) = 0\}$

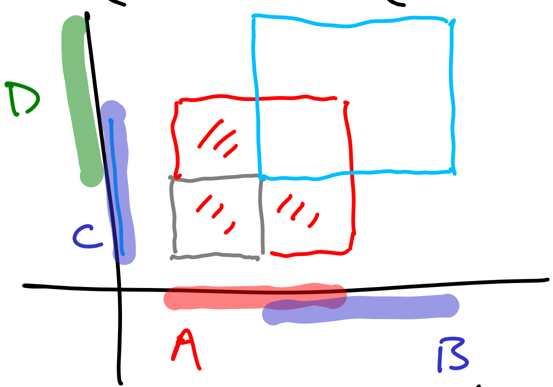
Correct: $\{x \in \mathbb{R} : f''(x) = 0 \wedge \forall \varepsilon > 0$

$\exists \delta > 0 \quad \delta < \varepsilon \quad f''(x-\delta) > 0$

$\wedge \; f''(x+\delta) < 0 \quad \vee$

$f''(x-\delta) < 0 \wedge f''(x+\delta) > 0\}$

(5) $(A\setminus B)\times(C\setminus D) \subseteq (A\times C)\setminus(B\times D)$



Let $[x,y] \in (A\setminus B)\times(C\setminus D)$

Then $x\in A\setminus B \wedge y \in C\setminus D$

Then $x\in A$, $x\notin B$, $y\in C$, $y\notin D$

Since $x\in A$, $y\in C$, $[x,y]\in A\times C$

Since $x\notin B$ $[x,y]\notin B\times D$

$\therefore [x,y]\in (A\times C)\setminus(B\times D)$

Let $A=\{a,b\}$, $a\neq b$

$B=\{b\}$

$C=\{c,d\}$ $c\neq d$

$D=\{d\}$

$[b,c]$ should work:

Since $b\in A$, $c\in C$

$[b,c]\in A\times C$

Since $c\notin D$ $[b,c]\notin B\times D$

$[b,c]\in (A\times C)\setminus(B\times D)$

$b\notin A\setminus B$ so $[b,c]\notin (A\setminus B)\times(C\setminus D)$

(6) $f: X \to Y \quad A, B \subseteq X$

$f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$

Let $y \in f_*(A \cap B)$

$\exists x \in A \cap B \quad f(x) = y$

Then $x \in A \wedge x \in B$

So $y = f(x) \in f_*(A) \wedge y = f(x) \in f_*(B)$

So $y \in f_*(A) \cap f_*(B)$.

Example: Let $X = \{x, x'\}$ $x \neq x'$

And $Y = \{y\}$ and $f: X \to Y$ is

given by $f(x) = f(x') = y$.

Let $A = \{x\}$, $B = \{x'\}$

$A \cap B = \phi$, so $f_*(A \cap B) = \phi$

$f_*(A) = Y = f_*(B)$

$f_*(A) \cap f_*(B) = Y \neq \phi$.

(7) $f: X \to Y \quad C, D \subseteq Y$

$$f^*(C \cap D) = f^*(C) \cap f^*(D)$$

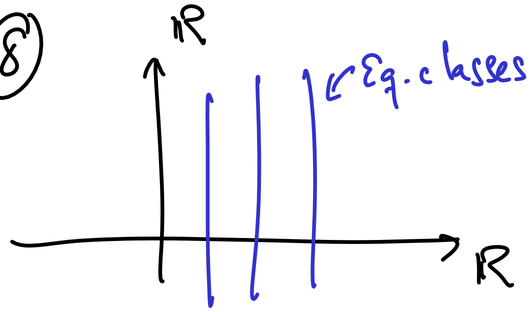
$x \in f^*(C \cap D) \iff f(x) \in C \cap D$

$\iff f(x) \in C \wedge f(x) \in D$

$\iff x \in f^*(C) \wedge x \in f^*(D)$

$\iff x \in f^*(C) \cap f^*(D)$

⑧



Note: $[x,y] \sim [x',y']$

$\iff x = x'$ ← $\pi_1([x',y'])$

↑ $\pi_1([x,y])$

Reflexive: $x = x$, so $[x,y] \sim [x,y]$

Symmetric: if $[x,y] \sim [x',y']$, then $x = x'$, so $x' = x$, so $[x',y'] \sim [x,y]$

Transitive: if $[x,y] \sim [x',y']$ and $[x',y'] \sim [x'',y'']$, $x = x'$ and $x' = x''$, so $x = x''$ so $[x,y] \sim [x'',y'']$

$[x,y] \sim [x',y'] \iff x = x'$, so equivalence classes are vertical lines.

⑨ Suppose $f : X \to Y$, $g : Y \to Z$
are function with $g \circ f$ onto.

Let $z \in Z$. Since $g \circ f$ is onto
$\exists x \in X$ $(g \circ f)(x) = z$

then $z = g(f(x))$ ☺

Let $X = \{x\}$, $Y = \{y, y'\}$ $y \neq y'$,

Define $f : X \to Y$ by $f(x) = y$

and $g : Y \to X$ by $g(y) = g(y') = x$

Then $g \circ f = Id|_X$ Since $(g \circ f)(x) = g(f(x)) = g(y) = x$

yet f is not onto, since y' is not in its range.

10 $S = \{x \in \mathbb{Q} : x^3 < 0\} = \{x \in \mathbb{Q} : x < 0\}$

1. S has no max: Pick $x \in S$.

Then $x < 0$, But $x < \frac{x}{2} < 0$, $\frac{x}{2} \in \mathbb{Q}$,

$\frac{x}{2} \in S$, so $x \neq \max S$

2. Claim $\sup S = 0$

If $x \in S$, then $x < 0$, so 0 is an upper bound for S.

Suppose $y < 0$, then by the density of $\mathbb{Q}$ in $\mathbb{R}$ $\exists\, a \in \mathbb{Q}$ $y < a < 0$

But $a \in S$, so y is not an upper bound for S.

3. If $y \in \mathbb{R}$, then $\exists\, a \in \mathbb{Q}$. $a < 0$, $a < y$. ($a \in 0 \cap y$)

then $a \in S$, so y is not a lower bound for S,

$\therefore$ S is not bdd below, so S has no min or inf ($\inf S = -\infty$)