① $P$ = plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$

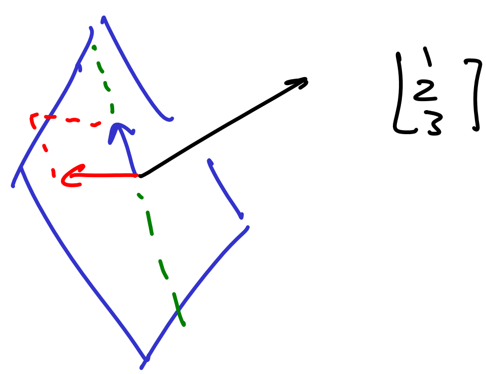
$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

a) Method 1 (geometry)

$\bar{0}$ is in $P$ (homog. eq.)

Suppose $\bar{u}$ $(\neq \bar{0})$ is $P$, then the line given by $\bar{u}$ and $\bar{0}$ is in $P$, so $P$ is closed under scalar mult.

Given $\bar{u}, \bar{v}$ in $P$, the parallelogram of $\bar{u}$ an $\bar{v}$ (and $\bar{0}$) is in $P$, so $P$ is closed under $+$.



Method 2 (algebra)

Again $1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0$ so $\bar{0}$ is in $P$

Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $P$ and a # $c$

$x + 2y + 3z = 0$, so $cx + 2cy + 3cz$

$= c(x + 2y + 3z) = c \cdot 0 = 0$

so $c \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $P$

$\therefore$ $P$ is closed under scalar mult.

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ are in $P$

Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + x' \\ y + y' \\ z + z' \end{bmatrix}$

$$x + x' + 2(y+y') + 3(z+z') = \underbrace{x + 2y + 3z}_{0} + \underbrace{x' + 2y' + 3z'}_{0} = 0$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ is in $P$, so

$P$ is closed under addition. ☺

Method 3: $P = \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ ☺

Method 4: $P = \underbrace{\operatorname{span} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\operatorname{Im} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{}^{\perp}$

b) $\begin{bmatrix} 1 & \underbrace{2 \;\; 3}_{} \end{bmatrix}$ free

$x + 2y + 3z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = y \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + z \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

Basis for $P$.

Gram - Schmidt

$$\left| \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right| = \sqrt{5}$$

$$\boxed{u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}$$

$$v_2 \cdot u_1 = \frac{6}{\sqrt{5}}$$

$$\operatorname{proj}_{u_1} v_2 = (v_2 \cdot u_1) u_1 = \frac{6}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2^{\perp} = v_2 - \operatorname{proj}_{u_1} v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix} \qquad |v_2^{\perp}| = \frac{1}{5}\sqrt{9 + 36 + 25}$$

$$= \frac{1}{5}\sqrt{70}$$

$$u_2 = \frac{v_2^{\perp}}{|v_2^{\perp}|} = \boxed{\frac{1}{\sqrt{70}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}}$$

② $P_1 \qquad \langle f, g\rangle = \int_0^1 fg\,dt$

Pick a basis for $P_1$ : $1, t$

$$\|1\| = \sqrt{\langle 1,1\rangle} = \sqrt{\int_0^1 dt} = 1$$

$$\langle t, 1\rangle = \int_0^1 t\,dt = \frac{1}{2}$$

$$\text{proj}_1 t = \langle t, 1\rangle \cdot 1 = \frac{1}{2}$$

$$t^\perp = t - \text{proj}_1 t = t - \frac{1}{2}$$

$$|t^\perp| = \sqrt{\int_0^1 (t-\tfrac{1}{2})(t-\tfrac{1}{2})\,dt}$$

$$= \sqrt{\frac{(t-\frac{1}{2})^3}{3}\Big|_0^1} = \sqrt{\frac{1}{24} - \left(-\frac{1}{24}\right)} = \frac{1}{3\sqrt{2}}$$

$$\frac{t^\perp}{|t^\perp|} = \boxed{3\sqrt{2}\left(t - \frac{1}{2}\right)}$$

3

a) $$\det \begin{bmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 5 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{bmatrix} =$$

Laplace

$$= -2 \det \begin{bmatrix} 2 & 3 & 2 \\ 6 & 5 & 3 \\ 7 & 0 & 4 \end{bmatrix}$$

$$= -2 \left[ 7 \underbrace{\det \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}}_{-1} + 4 \underbrace{\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}}_{-8} \right]$$

$$2(7+32) = 2 \cdot 39 = \boxed{78}$$

b)

1. $78 \neq 0$, so $A$ is invertible
2. $78 > 0$, so $A$ preserves orientation
3. $A$ scales 4-dimensional content (volume) by 78.

4

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$$

charpoly: $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix}$

$$= (2-\lambda)(-1-\lambda) - 4$$

$$= \lambda^2 - \lambda - 6 \qquad \lambda = 3, -2$$

$$A - 3I = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad x - 4y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ y \end{bmatrix} = y \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \bar{v}_1$$

$$A + 2I = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad x + y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \bar{v}_2$$

$$S = [\bar{v}_1 \ \bar{v}_2] = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

$$AS = \begin{bmatrix} 12 & 2 \\ 3 & -2 \end{bmatrix} \qquad S^{-1} A S = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$3v_1$ $\quad -2v_2$

b)