1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   (a) Find a basis for the kernel of A.

   (b) Find a basis for the image of A and sketch it.

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

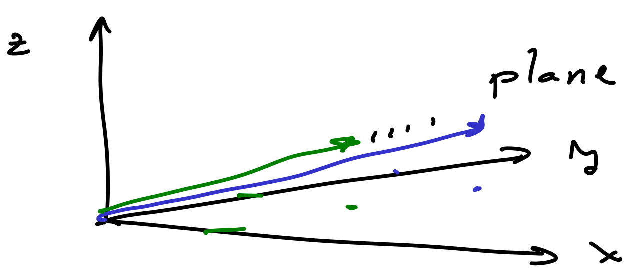
free

$$x_1 + x_2 - x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Basis for ker A

b) Basis for image: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ← a plane



2. Let $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Express $A\mathbf{v}_1$ and $A\mathbf{v}_2$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. What matrix represents the linear map $\mathbf{x} \mapsto A\mathbf{x}$ relative to the basis $[\mathbf{v}_1, \mathbf{v}_2]$?

Let $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \quad \underbrace{}_{S} \ \underbrace{}_{I}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} -1 & 2 \\ 1 & -1 \end{matrix}}_{S^{-1}} \end{bmatrix}$$

Then $S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$S^{-1}AS = \begin{bmatrix} 21 & 31 \\ -4 & -6 \end{bmatrix}$

$A v_1 = \begin{bmatrix} 13 \\ 17 \end{bmatrix} = 21 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A v_2 = \begin{bmatrix} 19 \\ 25 \end{bmatrix} = 31 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3. Let P_n be the vector space of all real polynomials $p(t)$ with degree $\le n$ and let Z_n be the set of all $p(t)$ in P_n such that $p(0) = 0$. Prove that

   (a) Z_n is a vector subspace of P_n.

   (b) T defined by $T(p(t)) = \int_0^t p(s)\,ds$ is a linear map from P_n to Z_{n+1}.

a) Let $z(t)$ be the zero polynomial.
Then $z(t) = 0$ for any t. In particular, $z(0) = 0$, so $z(t)$ is in Z_n

Suppose $p(t)$ and $q(t)$ are in Z_n, a, b are #s.

Then $p(0) = 0$, $q(0) = 0$, so

$(ap + bq)(0) = a\,p(0) + b\,q(0) = a \cdot 0 + b \cdot 0 = 0$

$\therefore$ $ap + bq$ is in Z_n ☺

b) If p is in P_n, then Tp is in Z_{n+1}

Let $p(t) = a_0 + a_1 t + \dots + a_n t^n$

$$Tp = \int_0^t p(s)\,ds = \left[a_0 s + a_1 \frac{s^2}{2} + \dots + a_n \frac{s^{n+1}}{n+1}\right]_0^t$$

$$= \underbrace{a_0 t + \dots + a_n \frac{t^{n+1}}{n+1}}_{\deg \le n+1 \text{ so in } P_{n+1}} - 0$$

plug in zero: Get zero so in Z_{n+1}

Given p, q in P_n and a, b #s

$$T(ap + bq) = \int_0^t [a\,p(s) + b\,q(s)]\,ds$$

$$\underset{\substack{\uparrow \\ \text{Sum rule}}}{=} \int_0^t a\,p(s)\,ds + \int_0^t b\,q(s)\,ds \underset{\substack{\uparrow \\ \text{const. multiple rule}}}{=} a\int_0^t p(s)\,ds + b\int_0^t q(s)\,ds = a\,Tp + b\,Tq$$ ☺

4. (c) Find the matrix for T with respect to bases $[1, t, t^2]$ for P_2 and $[t, t^2, t^3]$ for Z_3.

(d) Prove that T is invertible. Find a formula and the matrix for T^{-1}.

$$T(1) = t = 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3$$

$$T(t) = t^2/2 = 0 \cdot t + \tfrac{1}{2} t^2 + 0 \cdot t^3$$

$$T(t^2) = t^3/3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{matrix}}_{T^{-1}}$$

$$T^{-1}(t) = 1$$

$$T^{-1}(t^2) = 2t$$

$$T^{-1}(t^3) = 3t^2$$

$$T^{-1}(p) = \frac{dp}{dt}$$