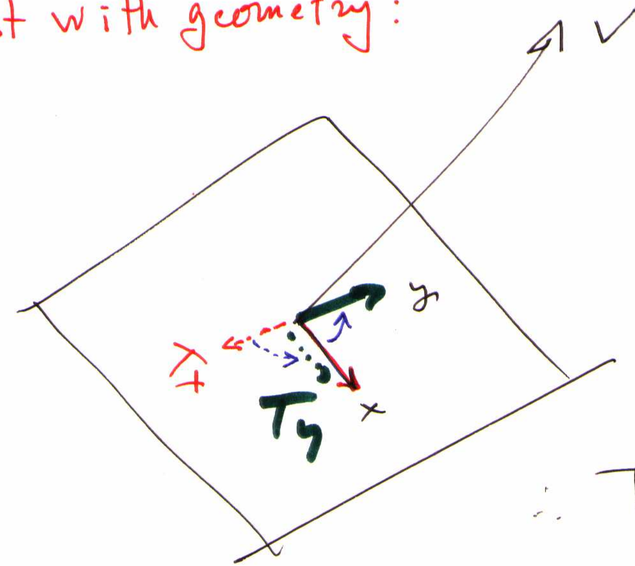
3 First with geometry:



$$T(x) = x \times v$$

∴ T ~~reverses~~ preserves orientation

$$|Tx| = |x \times v| = |x|\underbrace{|v|}_{\sqrt{3}}\underbrace{|\sin\theta|}_{1} = \sqrt{3} \qquad \therefore \det T > 0$$

Similarly $|Ty| = \sqrt{3}$

Area of square given by $\bar{x}$ & $\bar{y}$ $= 1$

Area of square given by $T\bar{x}$ & $T\bar{y}$ $= \sqrt{3}\sqrt{3} = 3$ $\quad \therefore |\det T| = 3$

$$\therefore \det = +3$$

Algebra:

Pick a Basis for the plane

$$x_1 + x_2 + x_3 = 0$$

e.g. $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{u_1} \quad \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_2}$

$$u_1 \times V = \text{Subdets}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$u_2 \times V = \text{Subdets}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

$\det A = +3$ ☺

(4) a) $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{bmatrix}$$

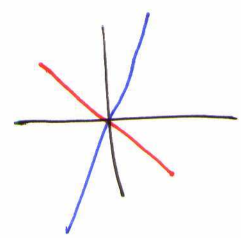
$$= (1-\lambda)(-1-\lambda) - 8 = \lambda^2 - 1 - 8 = \lambda^2 - 9$$
$$= (\lambda - 3)(\lambda + 3)$$

$\therefore$ eigenvalues of A are $3, -3$

$\lambda = 3$ $\quad A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$

$$\text{rref}(A - 3I) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x - \frac{1}{2} y = 0$ $\quad \leftarrow$ eigenspace



Pick eigenvector: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -3$ $\quad A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$

$$\text{rref}(A + 3I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x + y = 0$ $\quad \leftarrow$ eigenspace

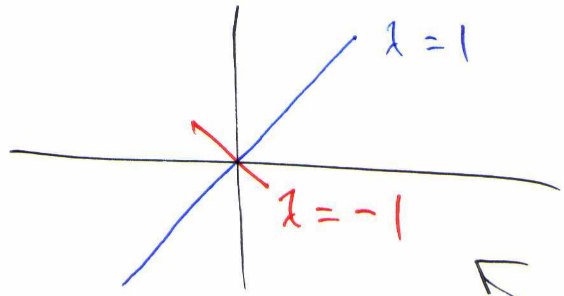
Pick eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, then $S^{-1} A S = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

eigen basis

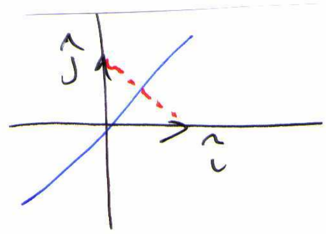
b) First with geometry

Reflection w.r.t. main diagonal



By inspection: eigenspaces

Now with algebra:



$\hat{i} \longmapsto \hat{j}$

$\hat{j} \longmapsto \hat{i}$

$\therefore\ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Check: $A\bar{u} = 2\,\text{proj}_{\begin{bmatrix}1\\1\end{bmatrix}}\bar{u} - \bar{u}$

$$A\begin{bmatrix}1\\0\end{bmatrix} = 2\frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = 2\frac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\det[A - \lambda I] = \det\begin{bmatrix}-\lambda & 1 \\ 1 & -\lambda\end{bmatrix} = \lambda^2 - 1 = (\lambda - \underline{1})(\lambda + \underline{1})$$

$$\text{rref}[A - I] = \text{rref}\begin{bmatrix}-1 & 1 \\ 1 & -1\end{bmatrix} = \begin{bmatrix}1 & -1 \\ 0 & 0\end{bmatrix} \quad \therefore\ \underline{x - y = 0}$$

$$\text{rref}[A + I] = \text{rref}\begin{bmatrix}1 & 1 \\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix} \quad \underline{x + y = 0}$$

Let $S = \begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$, then $S^{-1}AS = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}$

eigenbasis