MAT 2233 Linear Algebra Sp. 2002
Midterm 2

(1) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & \underbrace{0 \;\; 0}_{\text{Free}} & & 1 \end{bmatrix}$

pivot

Ker: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Basis for ker: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Basis for image: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

(2) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$\bar{x} \mapsto \bar{v} \times \bar{x}$, where $\bar{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

ker $= \{$ all $\bar{x} \uparrow\uparrow \bar{v} \}$ so a Basis for ker: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Image = Plane $\perp v$ $= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z = 0 \right\}$

A Basis for image: $\left\{ \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(3) Let $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $V = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

then U union V is not a subspace since it does not contain $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(any two lines through $\bar{0}$ will do)

(4) $\det[\bar{x}\ \bar{v}\ \bar{w}] = 0 \iff [\bar{x}\ \bar{v}\ \bar{w}]$ is not invertible

so $\ker T = \operatorname{span}\{\bar{v}\ \bar{w}\}$, $\dim(\ker T) = 2$

(This is a plane)

(5) $\det(\lambda I - A) = \lambda^2 - 5\lambda + 4$, Eigenvals: $4, 1$

$\lambda = 4 \Rightarrow \lambda I - A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\lambda = 1 \Rightarrow \lambda I - A = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

If $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ $\quad S^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$

$$A^n = S\begin{bmatrix} 4^n & 0 \\ 0 & 1 \end{bmatrix}S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4^n & 0 \\ 0 & 1 \end{bmatrix}\left(-\frac{1}{3}\right)\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -4^n & -4^n \\ -2 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -4^n-2 & -4^n+1 \\ -2\cdot 4^n+2 & -2\cdot 4^n-1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4^n+2 & 4^n-1 \\ 2\cdot 4^n-2 & 2\cdot 4^n+1 \end{bmatrix}$$