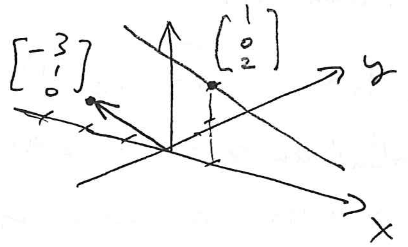
Midterm I   MAT 2233   Linear Algebra   Spring 2002

(1) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

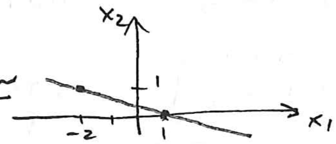
free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-3x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

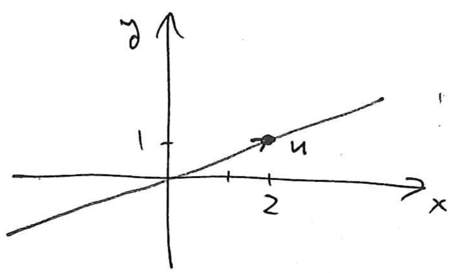
direction vector



Top view



(2)



Let $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

then $|u| = \sqrt{5}$

so $\hat{u} = \frac{u}{|u|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$T\bar{x} = (\hat{u} \cdot \bar{x})\,\hat{u} = \frac{1}{\sqrt{5}}(2x_1 + x_2)\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2}{5}x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus $A = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

(3) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Then $A\bar{x} = b$ has the unique solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(4) $A: \mathbb{R}^2 \to \mathbb{R}^3$ Since $A\bar{x} = 0 \Rightarrow \bar{x} = 0$, $\ker A = \{0\}$, i.e. A is 1-1.

$x \mapsto A \cdot x$

However A is not onto, since $\underset{\substack{\| \\ \operatorname{rank} A = 2}}{\dim(\text{image} A)} \neq \underset{\substack{\| \\ 3}}{\dim(\text{codomain})}$

If b is in the image A, then $A\bar{x} = \bar{b}$ has a unique solution.
If b is not in the image of A, then $A\bar{x} = \bar{b}$ has no solutions.

Here is a different explanation in terms of rref(A).
Since $A\bar{x} = 0$ has the unique solution $\bar{x} = 0$, both cols have leading 1s. Thus, one row has no leading 1.
Now look at the augmentation column, if we get a leading 1 there, no sols. If not, unique sol.

For example: $\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & x \\ 0 & 0 & ? \end{array}\right]$ Is this 0?
If 0, then unique sol.
If 1, then no sols.

(5)

$$T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Compute the inverse:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\text{So } T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$