1. (a) If P, Q, R are propositions, use a truth table to prove that
$(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$

The outputs in columns 5 and 8 below are the same.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge r$	$q \wedge r$	$p \wedge r \vee q \wedge r$
true	true	true	true	true	true	true	true
true	true	false	true	false	false	false	false
true	false	true	true	true	true	false	true
true	false	false	true	false	false	false	false
false	true	true	true	true	false	true	true
false	true	false	true	false	false	false	false
false	false	true	false	false	false	false	false
false	false	false	false	false	false	false	false

(b) If X, Y, Z are sets, prove that $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$

$x \in (X \cup Y) \cap Z$
$\Leftrightarrow x \in X \cup Y \wedge x \in Z$
$\Leftrightarrow (x \in X \vee x \in Y) \wedge x \in Z$
$\Leftrightarrow (x \in X \wedge x \in Z) \vee (x \in Y \wedge x \in Z)$ [by part (a)]
$\Leftrightarrow (x \in X \cap Z) \vee (x \in Y \cap Z)$
$\Leftrightarrow x \in (X \cap Z) \cup (Y \cap Z)$

2. Using formal language and appropriate quantifiers, translate into symbolic form the following sentences. Determine whether they are equivalent and explain why or why not.

- Some integers are not even and not odd.
  $\exists n \in \mathbf{Z}$ ($\sim n$ is even $\wedge \sim n$ is odd)
- Some integers are not even and some integers are not odd.
  ($\exists n \in \mathbf{Z}$ ($\sim n$ is even)) $\wedge$ ($\exists n \in \mathbf{Z}$ ($\sim n$ is odd))

The two sentences are not equivalent.

No integer is simultaneously not even and not odd, so the first sentence is false. [First sentence's negation $\forall n \in \mathbf{Z}$ (n is even $\vee$ n is odd) is true]

On the other hand there are integers that are not even (for example 1) and there are integers that are not odd (for example 0), so the second sentence is true.

3. For each statement below determine whether it is true. If so, prove it. If not, exhibit a concrete counterexample and explain why it is indeed a counterexample.

   (a) If a, b, c are integers and a divides bc, then a divides b or a divides c

   False.

   For example, let $a = 4$ and $b = c = 2$. Then $a = 4$ divides $bc = 4$, but $a = 4$ does not divide $b = 2$ and does not divide $c = 2$ ☹

   (b) If S and T are sets, $S \cup T = T \Leftrightarrow S \subseteq T$

   True.

   We need to prove $(S \cup T = T \Rightarrow S \subseteq T) \wedge (S \subseteq T \Rightarrow S \cup T = T)$

   For the first part let $x \in S$. Then $x \in S \cup T$, so if $S \cup T = T$, then $x \in T$ ☺

   Second part: to show $S \cup T = T$ we need to prove $(T \subseteq S \cup T) \wedge (S \cup T \subseteq T)$

   If $x \in T$, then $x \in S \cup T$, so $T \subseteq S \cup T$ ☺

   It remains to show $S \cup T \subseteq T$

   If $x \in S \cup T$, then $x \in S \vee x \in T$

   If $x \in T$, done, so suppose $x \in S$. If $S \subseteq T$, $x \in T$ ☺