Midterm 2

1. $48x - 34y = 6$ $\quad \gcd(48,34)=2 \quad (2|6 \ \ddot\smile)$

cancel 2: $24x - 17y = 3$

Extended Euclid:

$24 = 17 + 7$ $\quad 1 = 7 - 2\cdot 3 = 7 - 2(17 - 2\cdot 7) = -2\cdot 17 + 5\cdot 7$

$17 = 2\cdot 7 + 3$ $\quad = -2\cdot 17 + 5(24 - 17) = 5\cdot 24 - 7\cdot 17$

$7 = 2\cdot 3 + 1$ $\quad = 5\cdot 24 + 7(-17)$ $\quad$ so $3 = 15\cdot 24 + 21(-17)$

General solution: $x = 15 - 17n$

$y = 21 - 24n$

2. $4x \equiv 10 \bmod 13$ $\rightarrow$ $2x \equiv 5 \bmod 13$

$6x \equiv 9 \bmod 11$ $\quad 2x \equiv 3 \bmod 11$

(i) $2\cdot 7 = 13 + 1$, so $x = 5\cdot 7 = 13\cdot 2 + 9 = 9 \bmod 13$

$2\cdot 6 = 11 + 1$, so $x = 3\cdot 6 = 11 + 7 = 7 \bmod 11$

(ii) $\exists y \in \mathbb{Z}$ $x = 9 + 13y$, so $9 + 13y \equiv 7 \bmod 11$,

$2y \equiv -2 \bmod 11$, i.e. $y \equiv -1 \bmod 11$, i.e. $\exists z$ $y = -1 + 11z$

so $x = 9 + 13(-1 + 11z) = -4 + 143z$,

so $x \equiv -4 \bmod 143 \equiv 139 \bmod 143$

Alternate (ii): Chinese remainder formula: $x \equiv a_1 b_1 M_1 + a_2 b_2 M_2$

where $M = m_1 m_2 = 143$, $M_i = M/m_i$, $M_i b_i \equiv 1 \bmod m_i$

i	m_i	M_i	b_i	a_i	$a_i b_i M_i \bmod M$
1	13	11	6	9	$594 = 22 \bmod 143$
2	11	$\underbrace{13}_{2 \bmod 11}$	6	7	117

$6\cdot 11 = 5\cdot 13 + 1$

$2\cdot 6 = 11 + 1$

Sum: $x \equiv 139 \bmod 143$

3. If $x_0 = 2$, $x_1 = 4$, $\forall n > 1$ $x_n = 4x_{n-1} - 3x_{n-2}$,

then $\forall n \geq 0$ $x_n = 3^n + 1$

Proof by induction on n

Basis: $n = 0$ $x_0 = 3^0 + 1 = 2$ ☺

$n = 1$ $x_1 = 3^1 + 1 = 4$ ☺

Inductive step: Let $n > 1$

For all $k < n$ assume $x_k = 3^k + 1$

Then $x_n = 4(3^{n-1} + 1) - 3(3^{n-2} + 1) =$

$= 4 \cdot 3^{n-1} + 4 - 3^{n-1} - 3 = 3 \cdot 3^{n-1} + 1 = 3^n + 1$ ☺

4. $8^{1/7} = \sqrt[7]{8} \notin \mathbb{Q}$

If p is a prime, let $m(p, x)$ denote the multiplicity of p in the prime factorization of x.

If $8^{1/7} \in \mathbb{Q}$, $\exists r, s \in \mathbb{Z}$, $s \neq 0$ $8^{1/7} = \frac{r}{s}$.

so $s\,8^{1/7} = r$, so $s^7 8 = r^7$, so $m(2, s^7 8) = m(2, r^7)$

so $7m(2,s) + 3 = 7m(2,r)$, so $7 \mid 3$ ☹

Alt. proof: In the above we may assume $\gcd(r,s) = 1$

Since $s^7 8 = r^7$, $2 \mid r^7$ so by Euclid's lemma $2 \mid r$,

so $2^7 \mid s^7 8$, so $2 \mid s^7$, so by Euclid's lemma $2 \mid s$,

so both r and s are even ☹