Midterm 1 / 2022.9.22 / MAT 1313.001 / Algebra and number systems

1. (a) If P, Q, R are propositions, use a truth table to prove that
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(b) If X, Y, Z are sets, prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

(a)

P	Q	R	Q ∧ R	P ∨ Q ∧ R	P ∨ Q	P ∨ R	(P ∨ Q) ∧ (P ∨ R)
false	false	false	false	false	false	false	false
false	false	true	false	false	false	true	false
false	true	false	false	false	true	false	false
false	true	true	true	true	true	true	true
true	false	false	false	true	true	true	true
true	false	true	false	true	true	true	true
true	true	false	false	true	true	true	true
true	true	true	true	true	true	true	true

same outputs :)

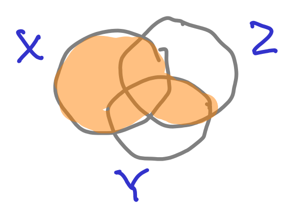
(b) $x \in X \cup (Y \cap Z) \Leftrightarrow x \in X \vee x \in Y \cap Z$

$\Leftrightarrow x \in X \vee (x \in Y \wedge x \in Z)$

$\Leftrightarrow (x \in X \vee x \in Y) \wedge (x \in X \vee x \in Z)$ (by part (a))

$\Leftrightarrow x \in X \cup Y \wedge x \in X \cup Z$

$\Leftrightarrow x \in (X \cup Y) \cap (X \cup Z)$:)



2. Using formal language and appropriate quantifiers, translate into symbolic form the following sentences. Determine whether they are equivalent and explain why or why not.

- Every integer is even or odd.
- Every integer is even or every integer is odd.

$\forall x \in \mathbb{Z}$ (x is even $\vee$ x is odd) (true)

$\forall x \in \mathbb{Z}$ $[(\exists n \in \mathbb{Z}\ x = 2n) \vee (\exists n \in \mathbb{Z}\ x = 2n+1)]$ $\left\{\begin{array}{l} 0 \le \text{rem.} < 2 \\ \text{so rem.} = 0 \\ \text{or rem.} = 1 \end{array}\right\}$

$(\forall x \in \mathbb{Z}$ x is even$) \vee (\forall x \in \mathbb{Z}$ x is odd$)$ (false $\vee$ false = false)

$(\forall x \in \mathbb{Z}\ \exists n \in \mathbb{Z}\ x = 2n) \vee (\forall x \in \mathbb{Z}\ \exists n \in \mathbb{Z}\ x = 2n+1)$

Not equivalent: one is true, the other is false.

3. For each statement below determine whether it is true. If so, prove it. If not, exhibit a concrete counterexample and explain why it is indeed a counterexample.

   (a) If a, b, c are integers such that a divides b and b divides c, then a divides c.

   (b) If a, b, c are integers such that a divides c and b divides c, then ab divides c.

(a) True. Suppose $a \mid b$ and $b \mid c$.

Then $\exists q_1 \in \mathbb{Z}$ $b = aq_1$ and $\exists q_2 \in \mathbb{Z}$ $c = bq_2$,

so $c = a q_1 q_2$ so $a \mid c$ ☺

(b) False. Let $a = b = c = 2$. Then $2 \mid 2$, but $4 \nmid 2$ ☹

4. For each statement below determine whether it is true. If so, prove it. If not, exhibit a concrete counterexample and explain why it is indeed a counterexample.

   (a) If S and T are sets, $S \cup T = S \Leftrightarrow S = T$

   (b) If S and T are sets, $S \cap T = S \Leftrightarrow S \subseteq T$

(a) False. Let $S = \{\emptyset\}$, $T = \emptyset$. Then $S \cup T = S$ but $S \neq T$.

(b) True.

Suppose $S \cap T = S$. Let $x \in S$, then $x \in S \cap T$, so $x \in T$ ☺

Conversely suppose $S \subseteq T$.

If $x \in S \cap T$, then $x \in S$

If $x \in S$, then $x \in T$ so $x \in S \cap T$ ☺