Midterm 2 / 2022.4.8 / MAT 1313.002 / Algebra and number systems

1. Find all simultaneous solutions to the system of equations

$$3x \equiv 4 \bmod 8$$

$$3x \equiv 8 \bmod 7$$

$1 \bmod 7$

$3 \cdot 3 = 9 \equiv 1 \bmod 8$ $\quad$ $3 \cdot 4 = 12 \equiv 4 \bmod 8$ $\quad$ $x \equiv 4 \bmod 8$

$3 \cdot 5 = 15 \equiv 1 \bmod 7$ $\quad$ $5 \cdot 1 = 5 \bmod 7$ $\quad$ $x \equiv 5 \bmod 7$

$x = 4 + 8y = 5 + 7z$, so $8y = 1 + 7z$, so $y = 1 + 7(z-y)$,

so $x = 4 + 8(1 + 7(z-y)) = 12 + 56(z-y) \equiv 12 \bmod 56$

Alternatively use the Chinese remainder formula:

$x \equiv (M_1 y_1 b_1 + M_2 y_2 b_2) \bmod M$

where $m_1 = 8$, $m_2 = 7$, $M = m_1 m_2 = 8 \cdot 7 = 56$, $M_1 = \frac{M}{m_1} = 7$, $M_2 = \frac{M}{m_2} = 8$,

$y_1 \equiv -1 \bmod 8$ $(-7 \equiv 1 \bmod 8)$, $y_2 \equiv 1 \bmod 7$ $(8 \equiv 1 \bmod 7)$, $b_1 = 4$, $b_2 = 5$

$x \equiv 7 \cdot (-1) 4 + 8 \cdot 1 \cdot 5 = 12 \bmod 56$

Check: $3 \cdot 12 = 36 = 32 + 4 \equiv 4 \bmod 8$

$3 \cdot 12 = 36 = 35 + 1 \equiv 1 \bmod 7$

2. Fibonacci numbers f_n ($n \geq 0$) are defined recursively by $f_0 = 0$, $f_1 = 1$ and for $n > 1$

$$f_n = f_{n-1} + f_{n-2}$$

(a) Compute the Fibonacci numbers for $n \leq 10$

(b) Prove that $f_n < 2^n$ for all $n \geq 0$

(a)

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

(b) Proof by induction on n:

Basis: $n=0$: $f_0 = 0 < 1 = 2^0$, $n=1$ $f_1 = 1 < 2 = 2^1$ ☺

Inductive step: If $n > 1$, assume for $k < n$ $f_k < 2^k$

For $k = n-1$ we get $f_{n-1} < 2^{n-1}$, for $k = n-2$ we get $f_{n-2} < 2^{n-2}$, so

$$f_n = f_{n-1} + f_{n-2} < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$$ ☺

3. Let $z = \dfrac{1+5i}{1+i\sqrt{2}}$

(a) Simplify z

(b) Find the real and imaginary parts of z

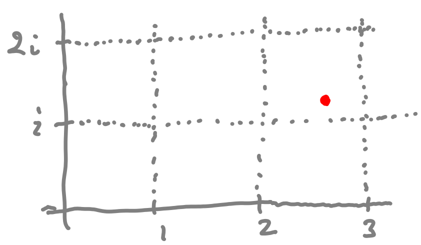
(c) Sketch z in the complex plane.

(d) Find $|z|$

(a) $z = \dfrac{1+5i}{1+i\sqrt{2}} \cdot \dfrac{1-i\sqrt{2}}{1-i\sqrt{2}} = \dfrac{1+5\sqrt{2}+i(5-\sqrt{2})}{3} = \dfrac{1+5\sqrt{2}}{3} + i\,\dfrac{5-\sqrt{2}}{3}$

(b) $\operatorname{Re} z = \dfrac{1+5\sqrt{2}}{3} \approx 2.7$, $\operatorname{Im} z = \dfrac{5-\sqrt{2}}{3} \approx 1.2$

(c)



(d) $|1+5i| = \sqrt{26}$, $|1+i\sqrt{2}| = \sqrt{3}$, so $|z| = \dfrac{|1+5i|}{|1+i\sqrt{2}|} = \dfrac{\sqrt{26}}{\sqrt{3}} \approx 3$