Midterm 1 / 2010.10.13 / MAT 1224.008 / Calculus II

Note Title 10/15/2010

1. Evaluate the following integrals (show all steps)

$$\text{(a)} \int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3} \qquad \text{(b)} \int \tanh\left(\frac{x}{5}\right)\, dx$$

a) Let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so $\frac{dx}{\sqrt{x}} = 2du$

$$2\int \frac{du}{u^3} = -\frac{1}{u^2} = \boxed{-\frac{1}{(1+\sqrt{x})^2} + C}$$

check

```
> 1/(sqrt(x)*(1+sqrt(x))^3); int(%,x);
```

$$\frac{1}{\sqrt{x}\,(1+\sqrt{x})^3}$$

$$-\frac{1}{(1+\sqrt{x})^2}$$

b) $\tanh\left(\frac{x}{5}\right) = \frac{\sinh\left(\frac{x}{5}\right)}{\cosh\left(\frac{x}{5}\right)}$ let $u = \cosh\left(\frac{x}{5}\right)$, then $du = \frac{1}{5}\sinh\left(\frac{x}{5}\right)dx$

so $\sinh\left(\frac{x}{5}\right)dx = 5\,du$

$$\int \frac{5\,du}{u} = 5\ln|u| = 5\ln\left|\cosh\left(\frac{x}{5}\right)\right| = \boxed{5\ln\left(\cosh\left(\frac{x}{5}\right)\right) + C}$$

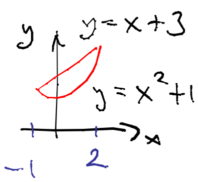
check:

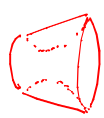
```
> sinh(x/5)/cosh(x/5); int(%,x);
```

$$\frac{\sinh\left(\frac{x}{5}\right)}{\cosh\left(\frac{x}{5}\right)}$$

$$5\ln\left(\cosh\left(\frac{x}{5}\right)\right)$$

2. Sketch the solid obtained by revolving the region in the plane enclosed by the curves $y = x^2+1$ and $y = x+3$ about the x axis. Find its volume.





$x^2+1 = x+3 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0$

$$V = \int_{-1}^{2} \left[\pi(x+3)^2 - \pi(x^2+1)^2\right] dx$$

$$= \pi\int_{-1}^{2}(x^2+6x+9-x^4-2x^2-1)\,dx = \pi\int_{-1}^{2}(-x^4-x^2+6x+8)\,dx$$

$$= \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + 3x^2 + 8x\right]_{-1}^{2}$$

$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + 6 - 8\right)\right] = \pi\left[-\frac{33}{5} + 30\right] = \boxed{\frac{117}{5}\pi \approx 73.5}$$

```
> f:=x->x^2+1; g:=x->x+3;
```

$$f := x \to x^2 + 1$$

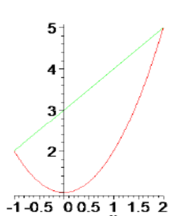
$$g := x \to x + 3$$

```
> solve(f(x)=g(x),x); xrange:=x=%[2]..%[1];
```

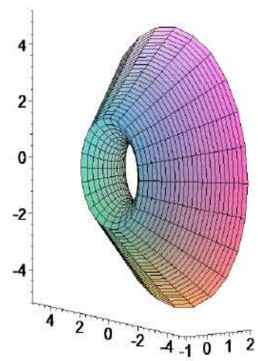
$$2, -1$$

$$xrange := x = -1 .. 2$$

```
> plot({f(x),g(x)},xrange,scaling=constrained);
```




```
> plot3d({[x,f(x)*cos(t),f(x)*sin(t)],[x,g(x)*cos(t),g(x)*sin(t)]}
,xrange,t=0..2*Pi,scaling=constrained,axes=frame);
```




```
> Pi*(g(x)^2-f(x)^2); dV:=expand(%); int(dV,x); int(dV,xrange);
evalf(%);
```

$$\pi\left((x+3)^2 - (x^2+1)^2\right)$$

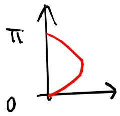
$$dV := -\pi x^2 + 6\pi x + 8\pi - \pi x^4$$

$$-\frac{1}{3}\pi x^3 + 3\pi x^2 + 8\pi x - \frac{1}{5}\pi x^5$$

$$\frac{117\pi}{5}$$

$$73.51326810$$

3. Sketch the surface obtained by rotating the curve segment $x = \sin y$, $0 \le y \le \pi$ about the y axis. Find its area.



$$dS = 2\pi x\sqrt{dx^2+dy^2} = 2\pi \sin y\sqrt{(\cos y\, dy)^2+dy^2}$$

$$= 2\pi\sin y\sqrt{(\cos y)^2+1}\; dy$$

Let $u=\cos y$, then $du = -\sin y\, dy$, so $S = \int_0^\pi dS = -2\pi\int\sqrt{u^2+1}\,du$

By parts: $\boxed{\int\sqrt{u^2+1}\,du} = u\sqrt{u^2+1} - \int u\,\underbrace{d\sqrt{u^2+1}}_{\frac{1}{\cancel{2}\sqrt{u^2+1}}\cdot\cancel{2}u\,du} = u\sqrt{u^2+1} - \int\frac{u^2}{\sqrt{u^2+1}}\,du$

$$= u\sqrt{u^2+1} - \int\frac{u^2+1-1}{\sqrt{u^2+1}}\,du$$

$$\boxed{= u\sqrt{u^2+1} - \int\sqrt{u^2+1}\,du + \int\frac{du}{\sqrt{u^2+1}}}$$

Solve for $\int\sqrt{u^2+1}\,du$:

$$\int\sqrt{u^2+1}\,du = \frac{1}{2}\left[u\sqrt{u^2+1} + \operatorname{arcsinh} u\right]$$

$$\therefore\ S = -\pi\left[\cos y\sqrt{(\cos y)^2+1} + \operatorname{arcsinh}(\cos y)\right]_0^\pi$$

$$= -\pi\left[\left(-\sqrt{2}+\operatorname{arcsinh}(-1)\right) - \left(\sqrt{2}+\operatorname{arcsinh}(1)\right)\right] = 2\pi\left[\sqrt{2}+\operatorname{arcsinh}(1)\right]$$

If $\theta = \operatorname{arcsinh}(1)$, $\sinh\theta = \frac{1}{2}\left[e^{\theta}-e^{-\theta}\right] = 1$, so $e^{\theta}-e^{-\theta}-2=0$

so $(e^{\theta})^2 - 2e^{\theta} - 1 = 0$, so $e^{\theta} = 1\pm\sqrt{2}$, but $e^{\theta}>0$, so $e^{\theta} = 1+\sqrt{2}$

so $\operatorname{arcsinh}(1) = \ln(1+\sqrt{2})$, so $\boxed{S = 2\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right] \simeq 14.4236}$

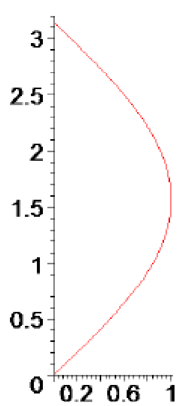
```
> f:=sin; yrange:=y=0..Pi;
```

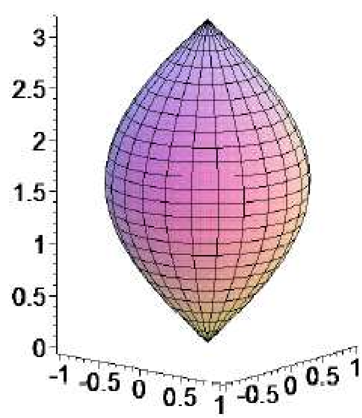
$$f := \sin$$

$$yrange := y = 0\,..\,\pi$$

```
> plot([f(y),y,yrange],scaling=constrained);

> plot3d([f(y)*cos(t),f(y)*sin(t),y],yrange,t=0..2*Pi,scaling=cons
  trained,axes=frame);
```






```
> dS:=2*Pi*f(y)*sqrt((diff(f(y),y))^2+1);
  int(dS,y); int(dS,yrange); evalf(%);
```

$$dS := 2\,\pi\,\sin(y)\sqrt{\cos(y)^2+1}$$

$$2\,\pi\left(-\frac{1}{2}\cos(y)\sqrt{\cos(y)^2+1} - \frac{1}{2}\operatorname{arcsinh}(\cos(y))\right)$$

$$2\,\pi\sqrt{2} + 2\,\pi\ln(1+\sqrt{2})$$

$$14.42359945$$

4. Luigi brews up a cup of espresso, which clocks at 90° (Celsius). He gets distracted by a phone call and after two minutes the espresso cools down to 60°. If the room temperature is 20°, how soon does Luigi need to hang up, if he doesn't want to drink tepid 30° espresso?

$\frac{dT}{dt} = k(T - T_0)$, let $u = T - T_0$, then $du = dT$, so $\frac{du}{dt} = ku$, so $u = Ae^{kt}$

so $T = T_0 + u = T_0 + Ae^{kt}$. Since $T_0 = 20$ and $T(0) = 90$, $T = 20 + 70e^{kt}$

Since $T(2) = 60$, $60 = 20 + 70e^{2k}$, $e^{2k} = \frac{40}{70} = \frac{4}{7}$, so $k = \frac{1}{2}\ln\frac{4}{7}$

so $\boxed{T(t) = 20 + 70\, e^{\frac{1}{2}\ln\frac{4}{7}t}}$. Now solve $30 = T(t)$ for t:

$30 = 20 + 70\, e^{\frac{1}{2}\ln\frac{4}{7}t}$, so $1 = 7e^{\frac{1}{2}\ln\frac{4}{7}t}$. Take ln:

$0 = \ln 7 + \frac{1}{2}\ln\frac{4}{7}t$ $\therefore$ $\boxed{t = -\frac{2\ln 7}{\ln\left(\frac{4}{7}\right)} \approx 7}$

$\therefore$ Luigi better hang up within the next 5 minutes

```
> diff(T(t),t)=k*(T(t)-20); ss:=dsolve({%,T(0)=90},T(t));
```

$$\frac{d}{dt}\mathrm{T}(t) = k\,(\mathrm{T}(t) - 20)$$

$$ss := \mathrm{T}(t) = 20 + 70\,e^{(k\,t)}$$

```
> subs(t=2,ss); subs(T(2)=60,%); solve(%,k); sss:=subs(k=%,ss);
```

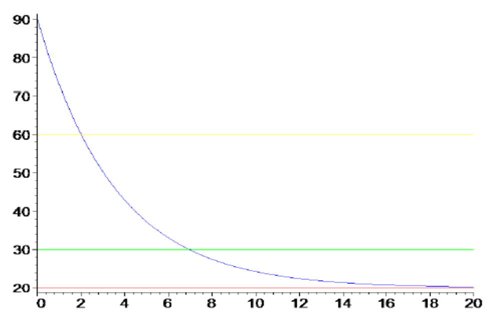
$$\mathrm{T}(2) = 20 + 70\,e^{(2\,k)}$$

$$60 = 20 + 70\,e^{(2\,k)}$$

$$\frac{1}{2}\ln\left(\frac{4}{7}\right)$$

$$sss := \mathrm{T}(t) = 20 + 70\,e^{(1/2\,\ln(4/7)\,t)}$$

```
> plot({subs(sss,T(t)),60,30,20},t=0..20);
```




```
> subs(T(t)=30,sss); solve(%,t); evalf(%);
```

$$30 = 20 + 70\,e^{(1/2\,\ln(4/7)\,t)}$$

$$-\frac{2\ln(7)}{\ln\left(\frac{4}{7}\right)}$$

$$6.954450502$$

5. Evaluate the following integrals (show all steps)

(a) $\int \arctan x\,dx$ (Hint: by parts) (b) $\int \frac{x^4\,dx}{1 - x^2}$

a) $\int \arctan x\,dx = x\arctan x - \int x\,\underbrace{d(\arctan x)}_{\frac{1}{1+x^2}dx}$

Let $u = 1 + x^2$, then $du = 2x\,dx$

So $x\,dx = \frac{1}{2}du$

So $\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|1 + x^2|$

$= \frac{1}{2}\ln(1 + x^2)$

$$\boxed{\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1 + x^2) + C}$$

b) $\frac{x^4}{1 - x^2} = \frac{x^4 - 1 + 1}{1 - x^2} = \frac{(x^2 - 1)(x^2 + 1) + 1}{1 - x^2} = -(x^2 + 1) + \frac{1}{1 - x^2}$

$= -x^2 - 1 - \frac{1}{(x - 1)(x + 1)}$ Since $\left.\frac{1}{x+1}\right|_{x=1} = \frac{1}{2}$, $\left.\frac{1}{x-1}\right|_{x=-1} = -\frac{1}{2}$

we get $-x^2 - 1 - \frac{1/2}{x - 1} + \frac{1/2}{x + 1}$

$$\boxed{\therefore \int \frac{x^4}{1 - x^2}dx = -\frac{x^3}{3} - x - \frac{1}{2}\ln|x - 1| + \frac{1}{2}\ln|x + 1| + C}$$

```
> arctan(x); int(%,x);
```

$$\arctan(x)$$

$$x\arctan(x) - \frac{1}{2}\ln(x^2 + 1)$$

```
> x^4/(1-x^2); int(%,x);
```

$$\frac{x^4}{1 - x^2}$$

$$-\frac{x^3}{3} - x - \frac{1}{2}\ln(x - 1) + \frac{1}{2}\ln(x + 1)$$