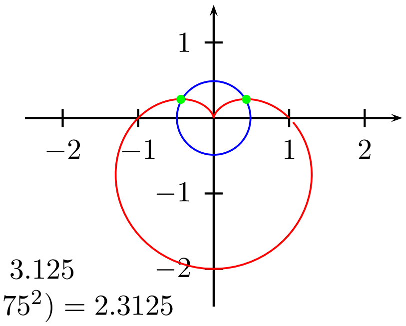
Midterm 1 solutions / 2006.3.9 / Calculus II / MAT 1223.005



1. $\frac{1}{2} = 1 - \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \ldots - \frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$

   $$\text{Area} = \frac{1}{2}\int_{-\frac{7\pi}{6}}^{\frac{\pi}{6}} \left[(1-\sin\theta)^2 - \left(\frac{1}{2}\right)^2\right] d\theta \approx 4.1335$$

2. Left $= 0.5(1^2 + 1.5^2) = 1.625$, Right $= 0.5(1.5^2 + 2^2) = 3.125$
   Trap $=$ (Left $+$ Right)$/2 = 2.375$, Mid $= 0.5(1.25^2 + 1.75^2) = 2.3125$
   Since x^2 is concave up, Mid is an underestimate and Trap is an overestimate.
   See the diagram on the next page.

3. For small x the dominant term in the numerator is 1, so this integral behaves like that of $x^{-\frac{4}{3}}$, which diverges by the p test with $p = \frac{4}{3} > 1$.

   Here is a comparison: $\displaystyle\int_0^1 \frac{\sqrt{\sqrt{x}+1}}{\sqrt[3]{x^4}}\,dx \geq \int_0^1 \frac{\sqrt{1}}{\sqrt[3]{x^4}}\,dx = \int_0^1 \frac{1}{x^{\frac{4}{3}}}\,dx = \infty$

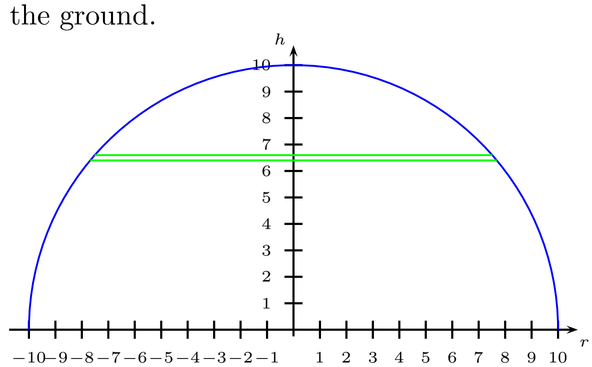
4. The equation for the half-circle that generates the hemisphere is $r^2 + h^2 = 10^2$.

   Density is a linear function of h with slope $(100-200)/10 = -10$, i.e. $\delta(h) = -10h + 200$.

   Since density is a function of only height, it is constant on horizontal planes, so the hemisphere will be sliced horizontally into discs. The mass of each slice at height h and thickness Δh is $\Delta m = \delta(h)\,\Delta V = \delta(h)\pi r^2\,\Delta h = \pi(200-10h)(10^2-h^2)\,\Delta h$. Summing the mass of all slices gives a Riemann sum, whose limit as $\Delta h \to 0$ is the Riemann integral for the total mass: $\pi\displaystyle\int_0^{10}(200-10h)(10^2-h^2)\,dh \approx 3.4\times 10^5$ kg.

   Vertical moment is the integral $\pi\displaystyle\int_0^{10} h(200-10h)(10^2-h^2)\,dh \approx 1.152\times 10^6$ kg·m

   The height of the center of mass is the ratio of vertical moment to total mass: 3.388 meters above the ground.



x^2	$(x+1)^{\frac{1}{2}}$
$2x$	$\frac{2}{3}(x+1)^{\frac{3}{2}}$
2	$\frac{4}{15}(x+1)^{\frac{5}{2}}$
0	$\frac{8}{105}(x+1)^{\frac{7}{2}}$

5. (a) Let $u = x+1$. Then $x = u-1$ and $du = dx$, so

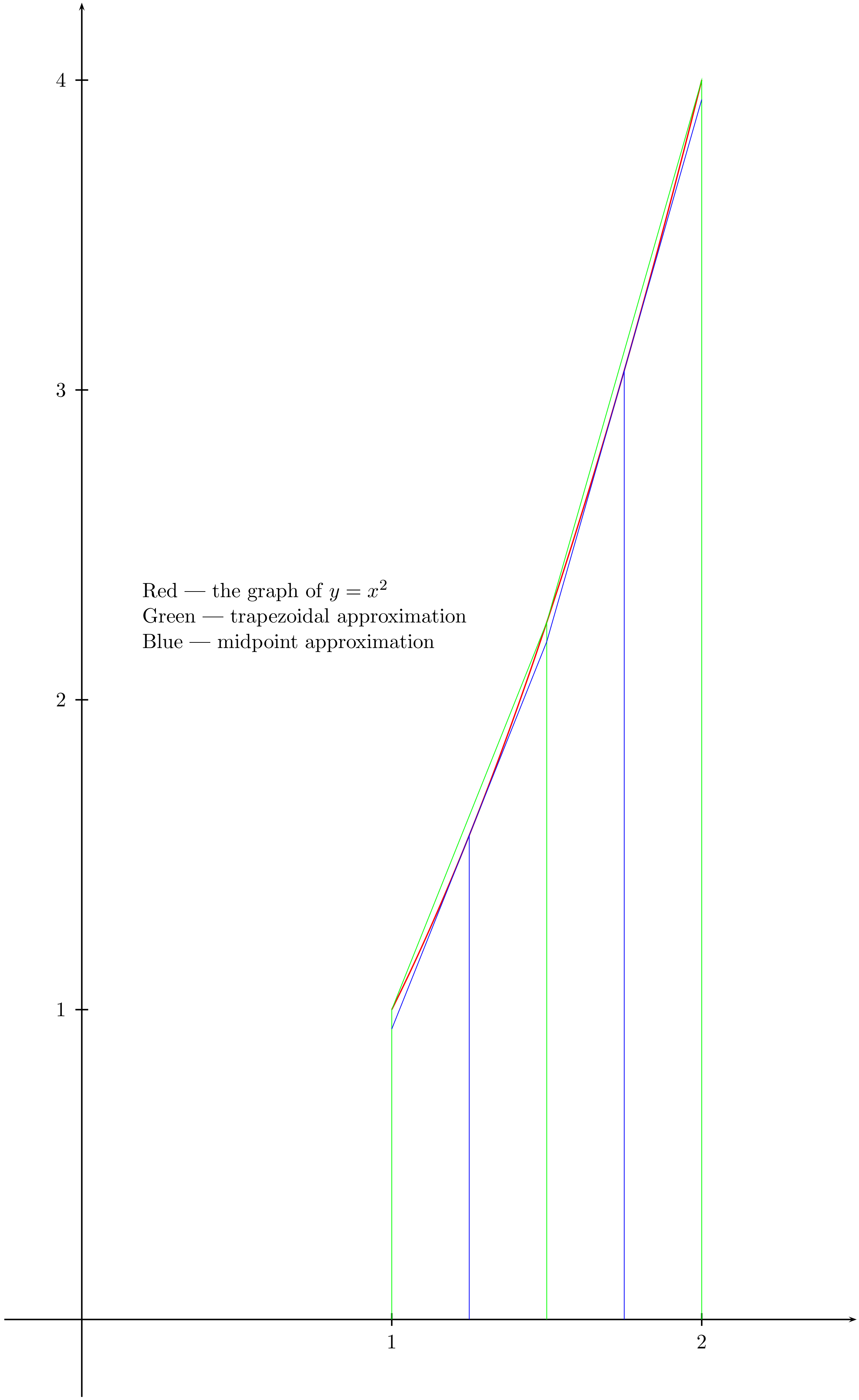
   $$\int x^2\sqrt{x+1}\,dx = \int (u-1)^2 u^{\frac{1}{2}}\,du = \int (u^2 - 2u + 1)u^{\frac{1}{2}}\,du = \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$$
   $$= \tfrac{2}{7}u^{\frac{7}{2}} - \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}} = \tfrac{2}{7}(x+1)^{\frac{7}{2}} - \tfrac{4}{5}(x+1)^{\frac{5}{2}} + \tfrac{2}{3}(x+1)^{\frac{3}{2}} = \tfrac{2}{105}(x+1)^{\frac{3}{2}}(15x^2 - 12x + 8) + C$$

   Alternately this integral can be evaluated by parts (see diagram above), to give

   $$\tfrac{2x^2}{3}(x+1)^{\frac{3}{2}} - \tfrac{8x}{15}(x+1)^{\frac{5}{2}} + \tfrac{16}{105}(x+1)^{\frac{7}{2}} = \tfrac{2}{105}(x+1)^{\frac{3}{2}}(15x^2 - 12x + 8) + C$$

   (b) Using long division and partial fractions gives

   $$\int \frac{x^2}{x^2-1}\,dx = \int\left(1 + \frac{1}{x^2-1}\right)dx = \int\left(1 + \frac{1}{(x-1)(x+1)}\right)dx$$
   $$= \int\left[1 + \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)\right]dx = x + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$


4
3
2
1
1
2
Red — the graph of $y = x^2$
Green — trapezoidal approximation
Blue — midpoint approximation