Midterm 2 / 2010.4.26 / Calculus for the Biosciences / MAT 1194.001

Note Title 4/26/2010

1. Find all critical points of $f(x) = x - x^3$ in the interval $-2 \le x \le 2$. Use f'' to determine whether they are local minima or maxima. Find the global minimum and maximum of f of the interval and state where they occur. Sketch.

$f' = 1 - 3x^2$

$f' = 0 \Rightarrow 3x^2 = 1, \; x^2 = \frac{1}{3}, \; x = \pm \frac{1}{\sqrt{3}}$

critical pts $\rightarrow \; \simeq \pm 0.577$

$f'' = -6x \qquad f''\left(\frac{1}{\sqrt{3}}\right) < 0 \quad \therefore$ local max

$f''\left(-\frac{1}{\sqrt{3}}\right) > 0 \quad \therefore$ local min

	x	$f(x)$	
crit	$-\frac{1}{\sqrt{3}}$	-0.385	
	$\frac{1}{\sqrt{3}}$	0.385	
end	-2	6	$\leftarrow$ max
	2	-6	$\leftarrow$ min

```
> f:=x-x^3;
```

$f := x - x^3$

```
> df:=diff(f,x);
```

$df := 1 - 3x^2$

```
> crit:=solve(df=0,x); evalf(%);
```

$crit := -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$

$-0.5773502693, 0.5773502693$

```
> ddf:=diff(df,x);
```

$ddf := -6x$

```
> [crit]; map(z->subs(x=z,ddf),%); evalf(%);
```

$\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$

$[2\sqrt{3}, -2\sqrt{3}]$

$[3.464101616, -3.464101616]$

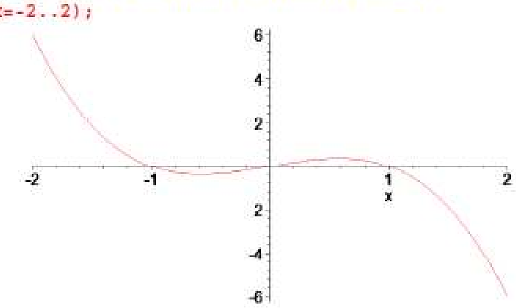
```
> [crit,-2,2]; map(z->subs(x=z,f),%); evalf(%);
```

$\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -2, 2\right]$

$\left[-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}, 6, -6\right]$

$[-0.3849001795, 0.3849001795, 6., -6.]$

```
> plot(f,x=-2..2);
```



2. Find indefinite integrals of the following functions

$$\text{(a) } \frac{e^{-2x}}{(1+e^{-2x})^2} \qquad \text{(b) } t^2\cos(3t)$$

a) let $u = 1 + e^{-2x}$, then $\frac{du}{dx} = e^{-2x}(-2)$

So $e^{-2x}\,dx = -\frac{1}{2}\,du$

$\therefore \int \frac{e^{-2x}}{(1+e^{-2x})^2}\,dx = -\frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}\frac{u^{-2+1}}{-2+1}$

$= \frac{1}{2}u^{-1} = \frac{1}{2}\frac{1}{u} = \frac{1}{2(1+e^{-2x})}$

```
> exp(-2*x)/(1+exp(-2*x))^2; int(%,x);
```

$\frac{e^{(-2x)}}{(1+e^{(-2x)})^2}$

$\frac{1}{2}\frac{1}{1+e^{(-2x)}}$

```
> t^2*cos(3*t); int(%,t);
```

$t^2\cos(3t)$

$\frac{1}{3}t^2\sin(3t) - \frac{2}{27}\sin(3t) + \frac{2}{9}t\cos(3t)$ $+C$

b)

t^2	$+$	$\cos(3t)$
$2t$	$-$	$\frac{1}{3}\sin(3t)$
2	$+$	$-\frac{1}{9}\cos(3t)$
0		$-\frac{1}{27}\sin(3t)$

3. Determine whether the improper integral $\int_1^\infty \frac{dx}{x^{\frac{2}{3}} + x^{\frac{4}{3}}}$ converges or diverges. Justify your assertion by comparison to an integral whose convergence or divergence can be determined directly.

For large x, $x^{4/3}$ dominates $x^{2/3}$, so compare to $\int_1^\infty \frac{dx}{x^{4/3}}$

Since $x^{2/3} + x^{4/3} > x^{4/3}$

$$0 \le \int_1^\infty \frac{dx}{x^{2/3} + x^{4/3}} \le \int_1^\infty \frac{dx}{x^{4/3}} = \int_1^\infty x^{-4/3}\,dx = \left.\frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}\right|_1^\infty = \left.-3x^{-\frac{1}{3}}\right|_1^\infty$$

$$= \left.-3\frac{1}{\sqrt[3]{x}}\right|_1^\infty = -3\left[0-1\right] = 3$$

$\therefore$ The original integral converges

(it's between 0 and 3)

```
> F:=1/(x^(2/3)+x^(4/3)); int(F,x); int(F,x=1..infinity);
  evalf(%);
```

$$F := \frac{1}{x^{(2/3)} + x^{(4/3)}}$$

$$3\arctan(x^{(1/3)})$$

$$\frac{3\pi}{4}$$

$$2.356194490$$

4. For the autonomous differential equation $dx/dt = x - a^3x^4$, where a is a positive constant, draw the phase-line diagram, find the equilibria, and determine their stability.

Equi: $\frac{dx}{dt} = 0$, $x - a^3x^4 = 0 \Rightarrow x(1-a^3x^3) = 0$

$x = 0$ or $a^3x^3 = 1$, $x^3 = \frac{1}{a^3}$, $x = \frac{1}{a}$

$x = 0$ is unstable

$x = \frac{1}{a}$ is stable

```
> f:=x-a^3*x^4;
```

$$f := x - a^3 x^4$$

```
> solve(f=0,x); equi:=[%[1],%[2]];
```

$$0, \frac{1}{a}, \frac{-1+\sqrt{3}\,I}{2a}, -\frac{1+\sqrt{3}\,I}{2a}$$

$$equi := \left[0, \frac{1}{a}\right]$$

```
> df:=diff(f,x);
```

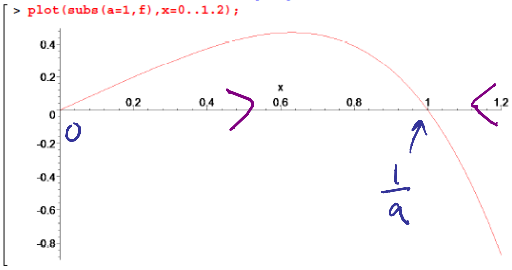
$$df := 1 - 4a^3x^3$$

```
> equi; map(z->subs(x=z,df),%);
```

$$\left[0, \frac{1}{a}\right]$$

$$[1, -3]$$

```
> plot(subs(a=1,f),x=0..1.2);
```



5. Solve the Torricelli differential equation $dh/dt = -\sqrt{h}$ with initial condition $h(0) = 5$. Sketch the solution and describe its long-term behavior.

$\frac{dh}{dt} = -\sqrt{h}$, $\frac{dh}{\sqrt{h}} = -dt$

$\int h^{-\frac{1}{2}}\,dh = -\int dt$, $\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -t + C$

$2\underbrace{h^{1/2}}_{\sqrt{h}} = -t + C$ I.C. $\Rightarrow 2\sqrt{5} = C$

$\therefore 2h^{1/2} = -t + 2\sqrt{5}$, $h^{1/2} = -\frac{t}{2} + \sqrt{5}$

$\boxed{h = \left(-\frac{t}{2} + \sqrt{5}\right)^2}$ As t increases to $2\sqrt{5} \approx 4.47$ $h(t)$ decreases to 0

```
> de:=diff(h(t),t)=-sqrt(h(t));
```

$$de := \frac{d}{dt}h(t) = -\sqrt{h(t)}$$

```
> dsolve(de,h(t));
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$$\sqrt{h(t)} + \frac{t}{2} - \_C1 = 0$$

```
> dsolve({de,h(0)=5},h(t)); ans:=subs(allvalues(%),h(t));
  evalf(%);
```

$$h(t) = \mathrm{RootOf}(-2\sqrt{\_Z} - t + 2\sqrt{5})$$

$$ans := \frac{1}{4}t^2 - t\sqrt{5} + 5$$

$$0.2500000000\,t^2 - 2.236067977\,t + 5.$$

```
> solve(ans=0,t); t1:=%[1]; evalf(%);
```

$$2\sqrt{5}, 2\sqrt{5}$$

$$t1 := 2\sqrt{5}$$

$$4.472135954$$

```
> plot(ans,t=0..t1);
```