Midterm 2 / 2008.12.2 / Calculus for the Biosciences / MAT 1194.002

Note Title 12/2/2008

1. Let $f(t) = t^4 - 2t^2$. Find all the critical points of f on the interval $-2 \le x \le 2$. Use the second derivative to determine concavity at the critical points. Find the global minimum and the global maximum of f on the interval. Where do they occur?

```
> f:=t^4-2*t^2;
```

$$f := t^4 - 2t^2$$

```
> df:=diff(f,t); factor(%);
```

$$df := 4t^3 - 4t$$

$$4t(t-1)(t+1)$$

```
> cp:=solve(df);
```

$cp := 0, 1, -1$ ← critical points

```
> ddf:=diff(df,t);
```

$$ddf := 12t^2 - 4$$

```
> [cp];map(xx->subs(t=xx,ddf),%);
```

$[0, 1, -1]$ ← f'' @ critical pts.

$[-4, 8, 8]$

f is concave down @ 0 and up @ 1, -1

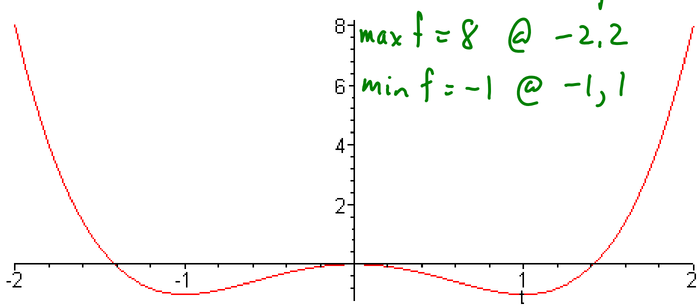
```
> [cp,-2,2]; map(xx->subs(t=xx,f),%);
```

endpoints

$[0, 1, -1, -2, 2]$

$[0, -1, -1, 8, 8]$ ← values of f

```
> plot(f,t=-2..2);
```



max $f = 8$ @ $-2, 2$

min $f = -1$ @ $-1, 1$

2. Find indefinite integrals of the following functions

(a) $e^{2t}(1 + e^{2t})^5$ (b) $t\cos(2t)$

```
> exp(2*t)*(1+exp(2*t))^5; int(%,t);
```

$$\int e^{(2t)}(1+e^{(2t)})^5 \, dt$$

$$= \frac{1}{12}(1+e^{(2t)})^6 + C$$

Let $u = e^{2t}$, $du = 2e^{2t}\,dt$

$e^{2t}\,dt = \frac{1}{2}\,du$

we get $\frac{1}{2}\int u^5\,du = \frac{1}{2}\frac{u^6}{6}$

```
> t*cos(2*t); int(%,t);
```

$$\int t\cos(2t)\,dt$$

$$= \frac{1}{4}\cos(2t) + \frac{1}{2}t\sin(2t) + C$$

t	$\oplus$	$\cos(2t)$
1	$\ominus$	$\frac{1}{2}\sin(2t)$
0		$-\frac{1}{4}\cos(2t)$

Long way: $\int t\cos(2t)\,dt = \int t\left(\frac{\sin(2t)}{2}\right)'\,dt = \frac{t\sin(2t)}{2} - \int \frac{\sin(2t)}{2}\,dt$

3. Show that the improper integral $\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converges and find an upper bound.

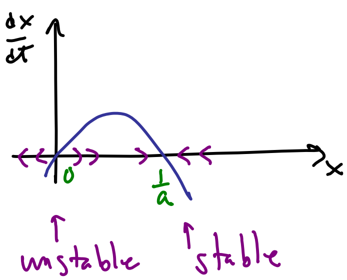
$$\sqrt{x} > 0 \;,\quad \sqrt{x} + x^2 > x^2 \;,\quad \frac{1}{\sqrt{x}+x^2} < \frac{1}{x^2}$$

$$\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^\infty = -x^{-1}\Big|_1^\infty$$

$$= -\frac{1}{x}\Big|_1^\infty = -0 + 1 = 1$$

4. For the autonomous differential equation $dx/dt = x - ax^2$, where a is a positive constant, draw the phase-line diagram, find the equilibria, and determine their stability.

$$\frac{dx}{dt} = x - ax^2 = x(1-ax) = 0 \quad \text{when} \quad x = 0 \text{ or } x = \frac{1}{a} \leftarrow \text{Equilibria}$$



5. Solve the Torricelli equation $dh/dt = -\sqrt{h}$ with initial condition $h(0) = 1$. When is $h = 0$?

```
> eq:=diff(h(t),t)=-sqrt(h(t));
  ic:=h(0)=1;
```

$$eq := \frac{d}{dt}\mathrm{h}(t) = -\sqrt{\mathrm{h}(t)}$$

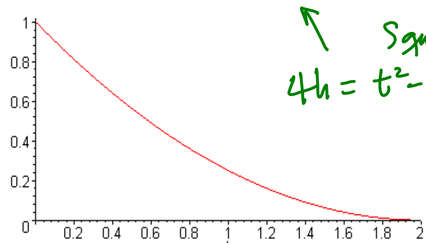
$$ic := \mathrm{h}(0) = 1$$

```
> dsolve({eq,ic},h(t)): allvalues(%);
  sol:=subs(%,h(t)): solve(%);
```

$$\mathrm{h}(t) = \frac{1}{4}t^2 - t + 1$$

2, 2

```
> plot(sol,t=0..2);
```



$$\int \frac{dh}{\sqrt{h}} = -\,dt \qquad \frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -t + C$$

$$\int h^{-\frac{1}{2}}\,dh = -t + C \qquad \frac{h^{1/2}}{1/2} = 2h^{\frac{1}{2}}$$

Since $h(0) = 1$, $C = 2$

$$2h^{1/2} = -t + 2$$

Square:

$$4h = t^2 - 4t + 4$$

Since $h^{1/2} \ge 0$

$t \le 2$

and

$h = 0$

$\Rightarrow t = 2$